

Binary Trees and Huffman Encoding

Computer Science E-22
Harvard University

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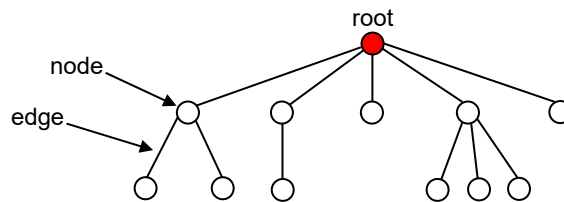
Motivation: Implementing a Dictionary

- A *data dictionary* is a collection of data with two main operations:
 - *search* for an item (and possibly delete it)
 - *insert* a new item
- If we use a *sorted* list to implement it, efficiency = $O(n)$.

<i>data structure</i>	<i>searching for an item</i>	<i>inserting an item</i>
a list implemented using an array	$O(\log n)$ using binary search	$O(n)$ because we need to shift items over
a list implemented using a linked list	$O(n)$ using linear search (binary search in a linked list is $O(n \log n)$)	$O(n)$ ($O(1)$ to do the actual insertion, but $O(n)$ to find where it belongs)

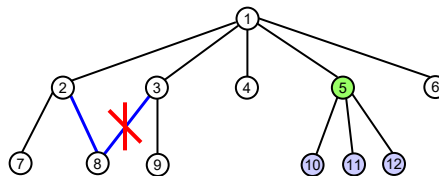
- In the next few lectures, we'll look at how we can use a *tree* for a data dictionary, and we'll try to get better efficiency.
- We'll also look at other applications of trees.

What Is a Tree?



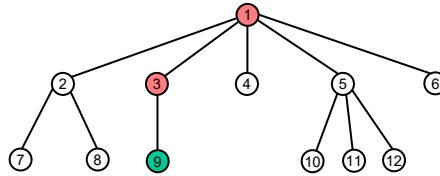
- A tree consists of:
 - a set of *nodes*
 - a set of *edges*, each of which connects a pair of nodes
- Each node may have one or more *data items*.
 - each data item consists of one or more fields
 - *key field* = the field used when searching for a data item
 - data items with the same key are referred to as *duplicates*
- The node at the "top" of the tree is called the *root* of the tree.

Relationships Between Nodes



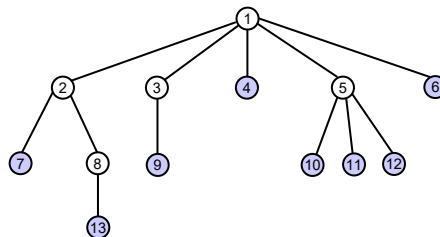
- If a node N is connected to nodes directly below it in the tree:
 - N is referred to as their *parent*
 - they are referred to as its *children*.
 - example: node 5 is the parent of nodes 10, 11, and 12
- Each node is the child of *at most one* parent.
- Nodes with the same parent are *siblings*.

Relationships Between Nodes (cont.)



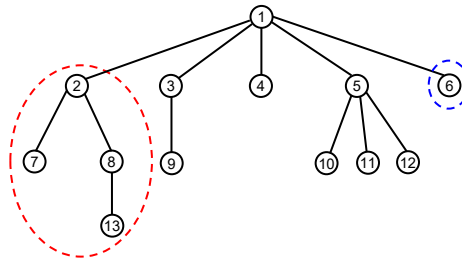
- A node's *ancestors* are its parent, its parent's parent, etc.
 - example: node 9's ancestors are 3 and 1
- A node's *descendants* are its children, their children, etc.
 - example: node 1's descendants are *all* of the other nodes

Types of Nodes



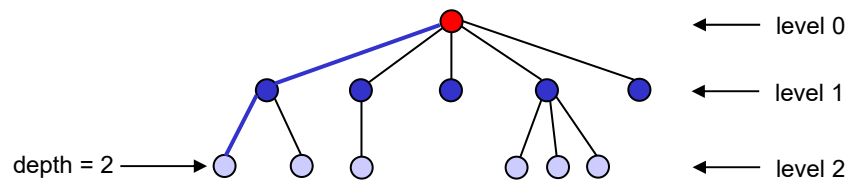
- A *leaf node* is a node without children.
- An *interior node* is a node with one or more children.

A Tree is a Recursive Data Structure



- Each node in the tree is the root of a smaller tree!
 - refer to such trees as *subtrees* to distinguish them from the tree as a whole
 - example: node 2 is the root of the subtree circled above
 - example: node 6 is the root of a subtree with only one node
- We'll see that tree algorithms often lend themselves to recursive implementations.

Path, Depth, Level, and Height

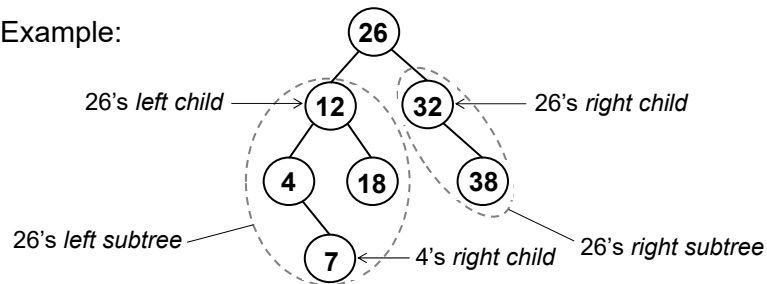


- There is exactly one *path* (one sequence of edges) connecting each node to the root.
- *depth* of a node = # of edges on the path from it to the root
- Nodes with the same depth form a *level* of the tree.
- The *height* of a tree is the maximum depth of its nodes.
 - example: the tree above has a height of 2

Binary Trees

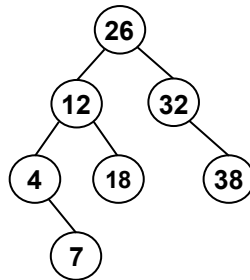
- In a *binary tree*, nodes have *at most two* children.
 - distinguish between them using the direction *left* or *right*

- Example:



- Recursive definition: a binary tree is either:
 - empty, or
 - a node (the root of the tree) that has:
 - one or more pieces of data (the key, and possibly others)
 - a *left subtree*, which is itself a binary tree
 - a *right subtree*, which is itself a binary tree

Which of the following is/are not true?



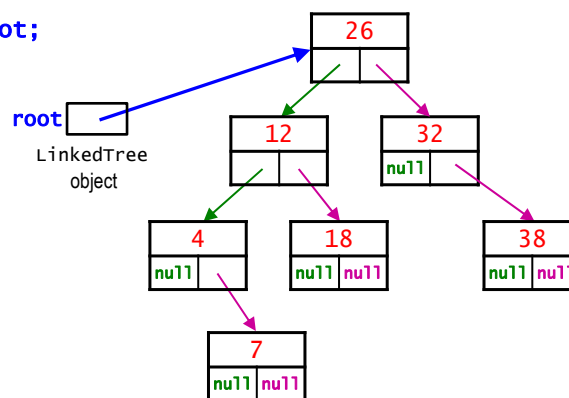
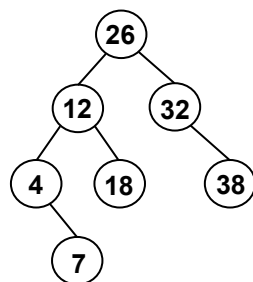
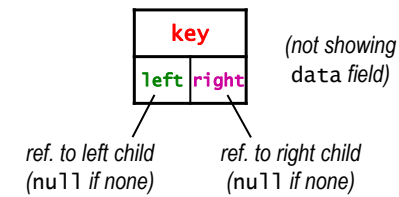
- This tree has a height of 4.
- There are 3 leaf nodes.
- The 38 node is the right child of the 32 node.
- The 12 node has 3 children.
- more than one of the above are not true (which ones?)

Representing a Binary Tree Using Linked Nodes

```
public class LinkedTree {
    private class Node {
        private int key;           // limit ourselves to int keys
        private LList data;        // list of data for that key
        private Node left;         // reference to left child
        private Node right;        // reference to right child
        ...
    }
    private Node root;
    ...
}
```

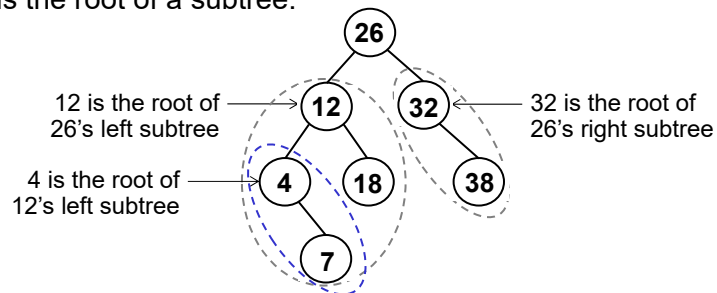
Representing a Binary Tree Using Linked Nodes

```
public class LinkedTree {
    private class Node {
        private int key;
        private LList data;
        private Node left;
        private Node right;
        ...
    }
    private Node root;
    ...
}
```



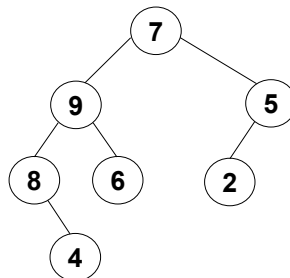
Traversing a Binary Tree

- Traversing a tree involves *visiting* all of the nodes in the tree.
 - visiting a node = processing its data in some way
 - example: print the key
- We'll look at four types of traversals.
 - each visits the nodes in a different order
- To understand traversals, it helps to remember that every node is the root of a subtree.



1: Preorder Traversal

- preorder traversal of the tree whose root is N:
 - 1) visit the root, N
 - 2) recursively perform a preorder traversal of N's left subtree
 - 3) recursively perform a preorder traversal of N's right subtree



- *preorder* because a node is visited *before* its subtrees
- The root of the tree as a whole is visited first.

Implementing Preorder Traversal

```
public class LinkedTree {
    ...
    private Node root;

    public void preorderPrint() {
        if (root != null) {
            preorderPrintTree(root);
        }
        System.out.println();
    }

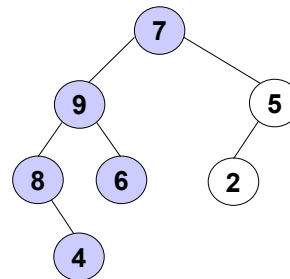
    private static void preorderPrintTree(Node root) {
        System.out.print(root.key + " ");
        if (root.left != null) {
            preorderPrintTree(root.left);
        }
        if (root.right != null) {
            preorderPrintTree(root.right);
        }
    }
}
```

*Not always the
same as the root
of the entire tree.*

- preorderPrintTree() is a static, recursive method that takes the root of the tree/subtree that you want to print.
- preorderPrint() is a non-static "wrapper" method that makes the initial call. It passes in the root of the entire tree.

Tracing Preorder Traversal

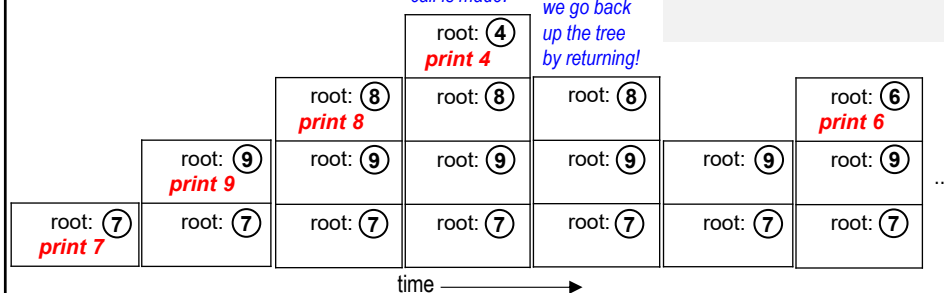
```
void preorderPrintTree(Node root) {
    System.out.print(root.key + " ");
    if (root.left != null) {
        preorderPrintTree(root.left);
    }
    if (root.right != null) {
        preorderPrintTree(root.right);
    }
}
```



order in which nodes are visited:

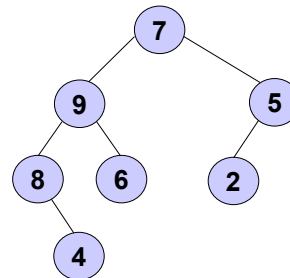
*base case, since
neither recursive
call is made!*

*we go back
up the tree
by returning!*



Using Recursion for Traversals

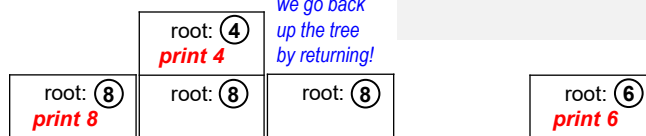
```
void preorderPrintTree(Node root) {
    System.out.print(root.key + " ");
    if (root.left != null) {
        preorderPrintTree(root.left);
    }
    if (root.right != null) {
        preorderPrintTree(root.right);
    }
}
```



*base case, since
neither recursive
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*we go back
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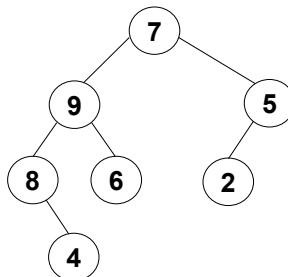
order in which nodes are visited:



- Using recursion allows us to easily go back up the tree.
- Using a loop would be harder. Why?

2: Postorder Traversal

- postorder traversal of the tree whose root is N:
 - 1) recursively perform a postorder traversal of N's left subtree
 - 2) recursively perform a postorder traversal of N's right subtree
 - 3) visit the root, N



- *postorder* because a node is visited *after* its subtrees
- The root of the tree as a whole is visited last.

Implementing Postorder Traversal

```
public class LinkedTree {
    ...
    private Node root;

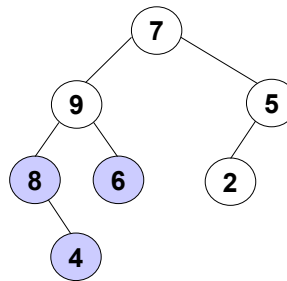
    public void postorderPrint() {
        if (root != null) {
            postorderPrintTree(root);
        }
        System.out.println();
    }

    private static void postorderPrintTree(Node root) {
        if (root.left != null) {
            postorderPrintTree(root.left);
        }
        if (root.right != null) {
            postorderPrintTree(root.right);
        }
        System.out.print(root.key + " ");
    }
}
```

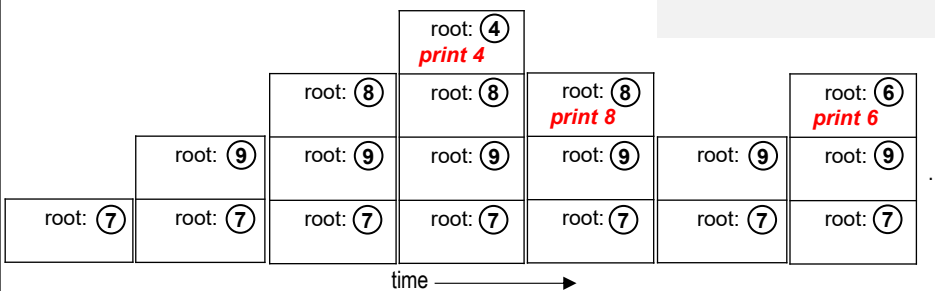
- Note that the root is printed *after* the two recursive calls.

Tracing Postorder Traversal

```
void postorderPrintTree(Node root) {
    if (root.left != null) {
        postorderPrintTree(root.left);
    }
    if (root.right != null) {
        postorderPrintTree(root.right);
    }
    System.out.print(root.key + " ");
}
```

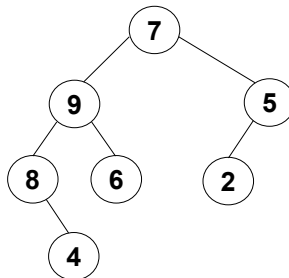


order in which nodes are visited:



3: Inorder Traversal

- inorder traversal of the tree whose root is N:
 - 1) recursively perform an inorder traversal of N's left subtree
 - 2) visit the root, N
 - 3) recursively perform an inorder traversal of N's right subtree



- The root of the tree as a whole is visited between its subtrees.
- We'll see later why this is called *inorder* traversal!

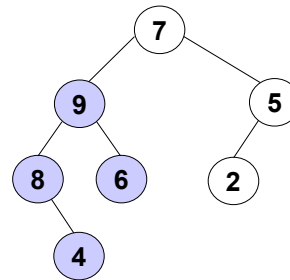
Implementing Inorder Traversal

```
public class LinkedTree {  
    ...  
    private Node root;  
    public void inorderPrint() {  
        if (root != null) {  
            inorderPrintTree(root);  
        }  
        System.out.println();  
    }  
    private static void inorderPrintTree(Node root) {  
        if (root.left != null) {  
            inorderPrintTree(root.left);  
        }  
        System.out.print(root.key + " ");  
        if (root.right != null) {  
            inorderPrintTree(root.right);  
        }  
    }  
}
```

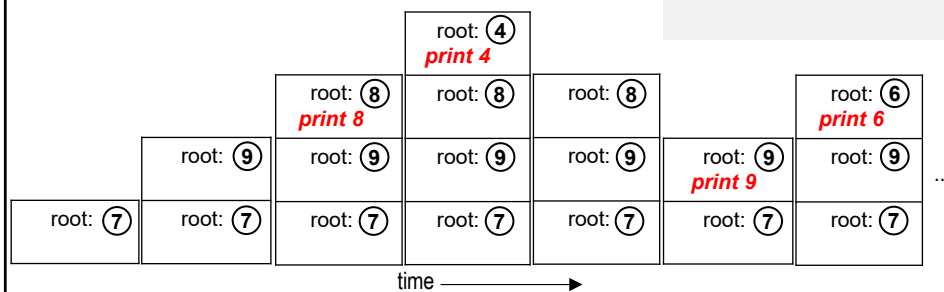
- Note that the root is printed *between* the two recursive calls.

Tracing Inorder Traversal

```
void inorderPrintTree(Node root) {
    if (root.left != null) {
        inorderPrintTree(root.left);
    }
    System.out.print(root.key + " ");
    if (root.right != null) {
        inorderPrintTree(root.right);
    }
}
```

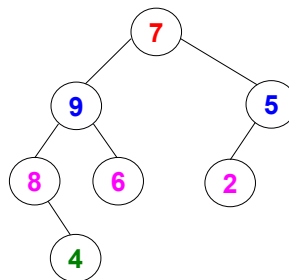


order in which nodes are visited:



Level-Order Traversal

- Visit the nodes one level at a time, from top to bottom and left to right.



- Level-order traversal of the tree above: **7 9 5 8 6 2 4**
- We can implement this type of traversal using a queue.

Tree-Traversal Summary

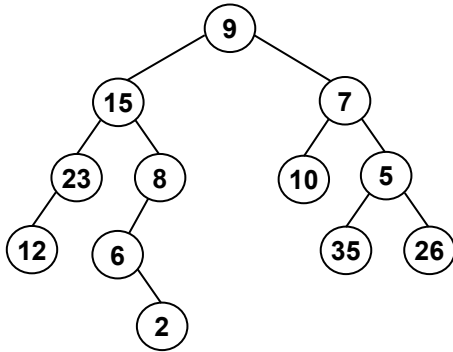
preorder: root, left subtree, right subtree

postorder: left subtree, right subtree, root

inorder: left subtree, root, right subtree

level-order: top to bottom, left to right

- Perform each type of traversal on the tree below:

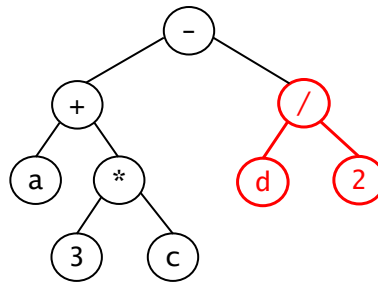


Tree Traversal Puzzle

- preorder traversal: A M P K L D H T
- inorder traversal: P M L K A H T D
- Draw the tree!
- What's one fact that we can easily determine from one of the traversals?

Using a Binary Tree for an Algebraic Expression

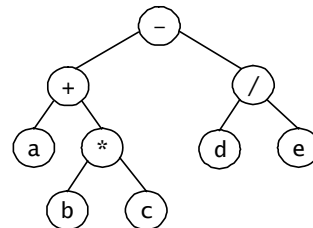
- We'll restrict ourselves to fully parenthesized expressions using the following binary operators: $+$, $-$, $*$, $/$
- Example: $((a + (3 * c)) - (d / 2))$



- Leaf nodes are variables or constants.
- Interior nodes are operators.
 - their children are their operands

Traversing an Algebraic-Expression Tree

- Inorder gives conventional algebraic notation.
 - print '(' before the recursive call on the left subtree
 - print ')' after the recursive call on the right subtree
 - for tree at right: $((a + (b * c)) - (d / e))$
- Preorder gives functional notation.
 - print '('s and ')'s as for inorder, and commas after the recursive call on the left subtree
 - for tree above: `subtr(add(a, mult(b, c)), divide(d, e))`
- Postorder gives the order in which the computation must be carried out on a stack/RPN calculator.
 - for tree above: push a, push b, push c, multiply, add,...



Fixed-Length Character Encodings

- A character encoding maps each character to a number.
- Computers usually use fixed-length character encodings.
 - ASCII - 8 bits per character

char	dec	binary
'a'	97	01100001
'b'	98	01100010
...	...	...
't'	116	01110100

example: "bat" is stored in a text file as the following sequence of bits:
01100010 01100001 01110100

- Unicode - 16 bits per character
(allows for foreign-language characters; ASCII is a subset)
- Fixed-length encodings are simple, because:
 - all encodings have the same length
 - a given character always has the same encoding

A Problem with Fixed-Length Encodings

- They tend to waste space.
- Example: an English newspaper article with only:
 - upper and lower-case letters (52 characters)
 - spaces and newlines (2 characters)
 - common punctuation (approx. 10 characters)
 - total of 64 unique characters → only need ____ bits
- We could gain even more space if we:
 - gave the most common letters shorter encodings (3 or 4 bits)
 - gave less frequent letters longer encodings (> 6 bits)

Variable-Length Character Encodings

- Variable-length encodings *compress* a text file by:
 - using encodings of different lengths for different characters
 - assigning shorter encodings to frequently occurring characters
- Example: if we had only four characters

e	01
o	100
s	111
t	00

"test" would be encoded as
00 01 111 00 → 000111100

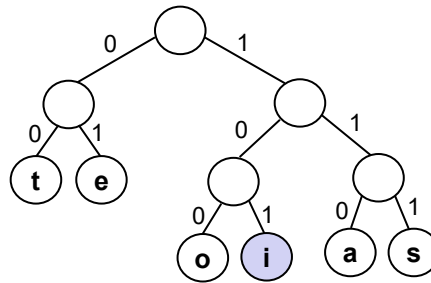
- Challenge: when reading a document, how do we determine the boundaries between characters?
 - how do we know how many bits the next character has?
- One requirement: no character's encoding can be the prefix of another character's encoding (e.g., couldn't have 00 and 001).

Huffman Encoding

- One type of variable-length encoding
- Based on the actual character frequencies in a given document
 - different documents have different encodings
- Huffman encoding uses a binary tree:
 - to determine the encoding of each character
 - to *decode* / *decompress* an encoded file
 - putting it back into ASCII

Huffman Trees

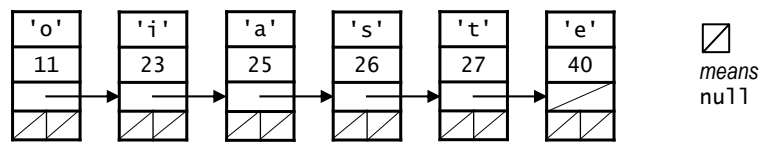
- Example for a text with only six characters:



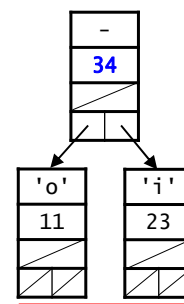
- Left branches are labeled with a 0, right branches with a 1.
- Leaf nodes are characters.
- To get a character's encoding, follow the path from the root to its leaf node.
 - example: i = ?

Building a Huffman Tree

- Begin by reading through the text to determine the frequencies.
- Create a list of nodes containing (character, frequency) pairs for each character in the text – *sorted by frequency*.

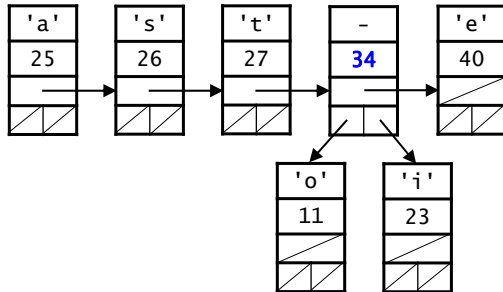


- Remove and "merge" the nodes with the two lowest frequencies, forming a new node that is their parent.
 - left child = lowest frequency node
 - right child = the other node
 - frequency of parent = sum of the frequencies of its children
 - in this case, $11 + 23 = 34$



Building a Huffman Tree (cont.)

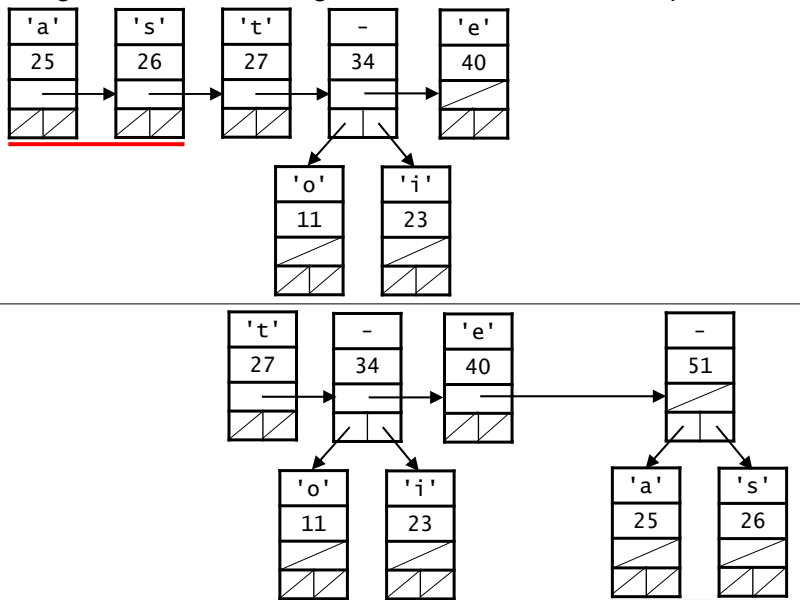
4) Add the parent to the list of nodes (maintaining sorted order):



5) Repeat steps 3 and 4 until there is only a single node in the list, which will be the root of the Huffman tree.

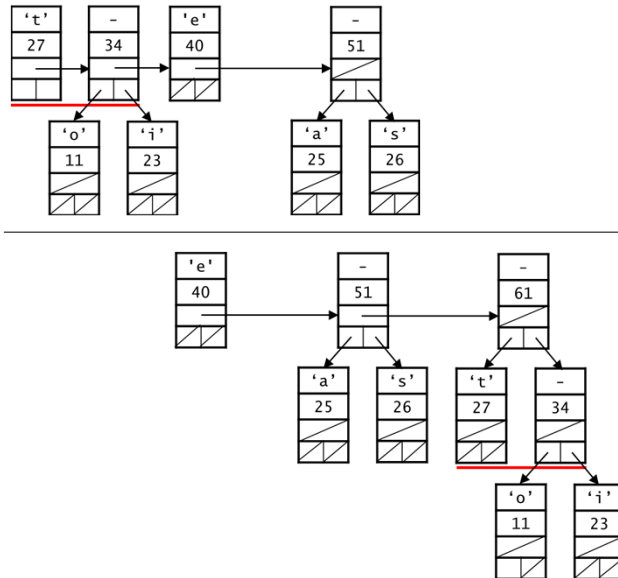
Completing the Huffman Tree Example I

• Merge the two remaining nodes with the lowest frequencies:



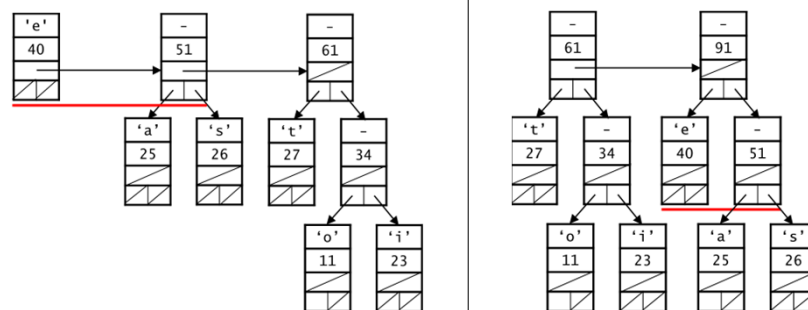
Completing the Huffman Tree Example II

- Merge the next two nodes:



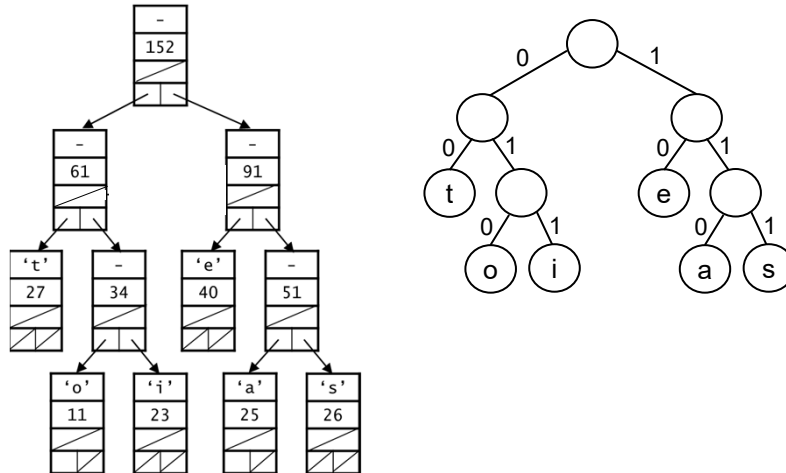
Completing the Huffman Tree Example II

- Merge again:



Completing the Huffman Tree Example IV

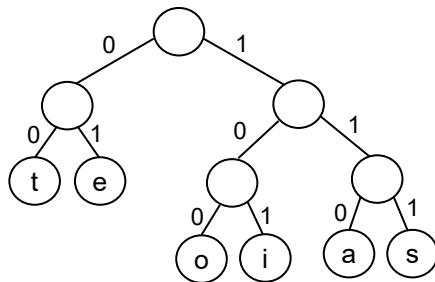
- The next merge creates the final tree:



- Characters that appear more frequently end up higher in the tree, and thus their encodings are shorter.

The Shape of the Huffman Tree

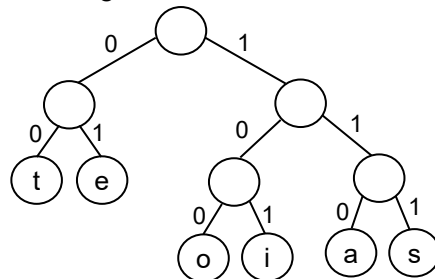
- The tree on the last slide is fairly symmetric.
- This won't always be the case!
 - depends on the character frequencies
- For example, changing the frequency of 'o' from 11 to 21 would produce the tree shown below:



- This is the tree that we'll use in the remaining slides.

Huffman Encoding: Compressing a File

- 1) Read through the input file and build its Huffman tree.
- 2) Write a file header for the output file.
 - include the character frequencies so the tree can be rebuilt when the file is decompressed
- 3) Traverse the Huffman tree to create a table containing the encoding of each character:



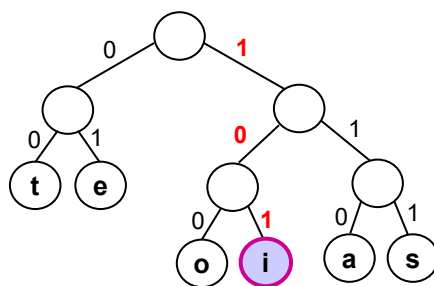
a	
e	
i	
o	
s	
t	

- 4) Read through the input file a second time, and write the Huffman code for each character to the output file.

Huffman Decoding: Decompressing a File

- 1) Read the frequency table from the header and rebuild the tree.
- 2) Read one bit at a time and traverse the tree, starting from the root:
 - when you read a bit of 1, go to the right child
 - when you read a bit of 0, go to the left child
 - when you reach a leaf node, record the character, return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!

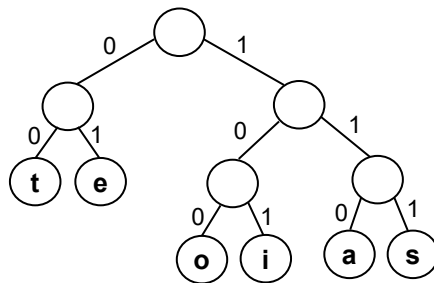


example: **1**0111110000111100
first character = i

What are the next three characters?

- 1) Read the frequency table from the header and rebuild the tree.
- 2) Read one bit at a time and traverse the tree, starting from the root:
 - when you read a bit of 1, go to the right child
 - when you read a bit of 0, go to the left child
 - when you reach a leaf node, record the character,
 - return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!

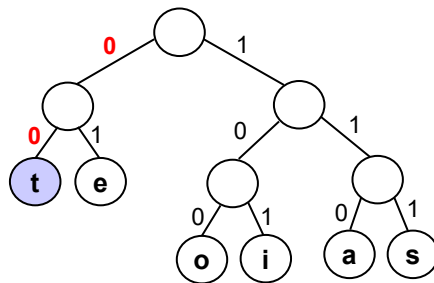


example: 101111110000111100
first character = i (101)

Huffman Decoding: Decompressing a File

- 1) Read the frequency table from the header and rebuild the tree.
- 2) Read one bit at a time and traverse the tree, starting from the root:
 - when you read a bit of 1, go to the right child
 - when you read a bit of 0, go to the left child
 - when you reach a leaf node, record the character,
 - return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!



example: 101111110000111100

101 = right, left, right = i
111 = right, right, right = s
110 = right, right, left = a
00 = left, left = t
01 = left, right = e
111 = right, right, right = s
00 = left, left = t