Computer Science E-22
Data Structures

Harvard Extension School, Fall 2025
David G. Sullivan, Ph.D.

Introduction: Course Overview, Java Review

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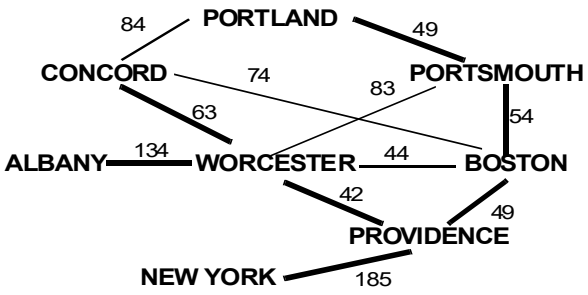
Welcome to Computer Science E-22/S-22!

- We will study fundamental *data structures.*
  - ways of imposing order on a collection of information
  - sequences: lists, stacks, and queues
  - trees
  - hash tables
  - graphs
- We will also:
  - study *algorithms* related to these data structures
  - learn how to *compare* data structures & algorithms
- Goals:
  - learn to think more intelligently about programming problems
  - acquire a set of useful tools and techniques

Sample Problem I: Finding Shortest Paths

- Given a set of routes between pairs of cities, determine the shortest path from city A to city B.



Sample Problem II: A Data "Dictionary"

- Given a large collection of data, how can we arrange it so that we can efficiently:
  - add a new item
  - search for an existing item
- Some data structures provide better performance than others for this application.
- More generally, we'll learn how to characterize the *efficiency* of different data structures and their associated algorithms.

Prerequisites

- A good working knowledge of Java
  - comfortable with object-oriented programming concepts
  - comfortable with arrays
  - some prior exposure to recursion would be helpful
  - if your skills are weak or rusty, you may want to consider first taking CSCI E-10b
- Reasonable comfort level with mathematical reasoning
  - mostly simple algebra, but need to understand the basics of logarithms (we'll review this)
  - will do some simple proofs

Requirements

- Lectures
- Sections
  - optional but highly recommended
  - start this week
  - on Zoom; one section will be recorded
- Five problem sets
  - plan on 10-20 hours per week!
  - code in Java
  - must be your own work
    - see syllabus for the collaboration policy
  - grad-credit students will do extra problems
- Midterm exam
- Final exam

Additional Administrivia

- Instructor: Dave Sullivan
- TAs: see website mentioned below
- Office hours and contact info. will be available on the Web:
  https://cscie22.sites.fas.harvard.edu
- For questions on content, homework, etc.:
  - use Ed Discussion on Canvas
  - send e-mail to cscie22-staff@lists.fas.harvard.edu

Review: What is an Object?

- An *object* groups together:
  - one or more data values (the object's *fields* – also known as *instance variables*)
  - a set of operations that the object can perform (the object's *methods*)
- In Java, we use a *class* to define a new type of object.
  - serves as a "blueprint" for objects of that type
  - simple example:

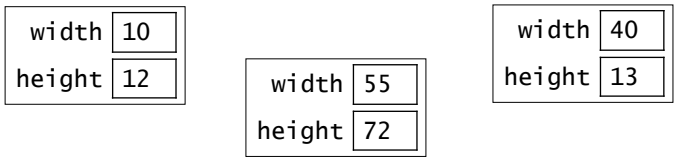
```
public class Rectangle {
    // fields
    private int width;
    private int height;

    // methods
    public int getWidth() {
        return this.width;
    }
    …
```

Class vs. Object

- The `Rectangle` class is a blueprint:

```
public class Rectangle {
    // fields
    private int width;
    private int height;

    // methods
    ...
}
```

- `Rectangle` objects are built according to that blueprint:



(You can also think of the methods as being inside the object, but we won't show them in our diagrams.)

Creating and Using an Object

- We create an object by using the `new` operator and a special method known as a *constructor*:

```
Rectangle r1 = new Rectangle(10, 30);
```

- Once an object is created, we can call one of its methods by using *dot notation*:

```
int width1 = r1.getWidth();
```

- The object on which the method is invoked is known as the *called object* or the *current object.*

Two Types of Methods

- Methods that belong to an object are referred to as *instance methods* or *non-static methods.*
  - they are invoked on an object

```
int width1 = r1.getWidth();
```

  - they have access to the fields of the called object
- *Static* methods do *not* belong to an object – they belong to the class as a whole.
  - they have the keyword static in their header:

```
public static int max(int num1, int num2) {
    ...
```

  - they do *not* have access to the fields of the class
  - outside the class, they are invoked using the class name:

```
int result = Math.max(5, 10);
```

Encapsulation

- Our classes should provide proper *encapsulation*.
- We limit direct access to the internals of an object by making the fields *private:*

```
public class Rectangle {
    private int width;
    private int height;
    ...
```

  - `private` components of a class can only be accessed directly by code within the class itself
- We provide limited *indirect* access through methods that are labeled *public.*

```
    public int getWidth() {
        return this.width;
    }
```

  - `public` components can be accessed anywhere

Encapsulation (cont.)

- `getWidth()` is an *accessor* method that can be used to obtain information about an object, but *not* to change it:

```
public int getWidth() {
    return this.width;
}
```

  - we use the keyword `this` to access the fields and methods that are inside the called object

- A class can also provide *mutator* methods that change the called object, but only in appropriate ways:

```
public void setWidth(int newWidth) {
    if (newWidth <= 0) {
        throw new IllegalArgumentException();
    }
    this.width = newWidth;
}
```

  - throwing an exception prevents an inappropriate change by ending the method prematurely

Inheritance

- We can define a class that explicitly *extends* another class:

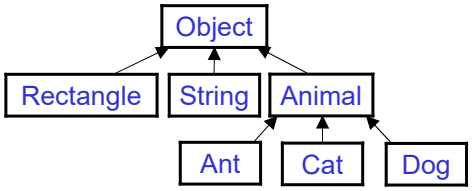
```
public class Animal {
    private String name;
    // other field definitions go here

    public String getName() {
        return this.name;
    }
    // other method definitions go here
}

public class Dog extends Animal {
    ...
```

- We say that `Dog` is a *subclass* of `Animal`, and `Animal` is a *superclass* of `Dog`.

- A class *inherits* the fields and methods of the class that it extends.

The `Object` Class

- If a class does not explicitly extend another class, it implicitly extends Java's `Object` class.
- The `Object` class includes methods that all classes must possess. For example:
  - `toString()`: returns a string representation of the object
  - `equals()`: is this object equal to another object?
- The process of extending classes forms a hierarchy of classes, with the `Object` class at the top of the hierarchy:



Polymorphism

- An object can be used wherever an object of one of its superclasses is called for.
- For example:

```
Animal a = new Dog();

Animal[] zoo = new Animal[100];
zoo[0] = new Ant();
zoo[1] = new Cat();
...
```

- The name for this capability is *polymorphism.*
  - from the Greek for "many forms"
  - the same code can be used with objects of different types

A Bag Data Structure

- A bag is just a container for a group of data items.
  - analogy: a bag of candy
- The positions of the data items don't matter (unlike a sequence).
  - {3, 2, 10, 6} is equivalent to {2, 3, 6, 10}
- The items do *not* need to be unique (unlike a set).
  - {**7**, 2, 10, **7**, 5} isn't a set, but it is a bag

A Bag Data Structure (cont.)

- The operations we want our bag to support:
  - `add(item)`: add `item` to the bag
  - `remove(item)`: remove one occurrence of `item` (if any) from the bag
  - `contains(item)`: check if `item` is in the bag
  - `numItems()`: get the number of items in the bag
  - `grab()`: get an item at random, without removing it
    - reflects the fact that the items don't have a position (and thus we can't say "get the 5th item in the bag")
  - `toArray()`: get an array containing the current contents of the bag

Implementing a Bag Using a Class

- To implement our bag data structure, we define a class:

```
public class ArrayBag {
    private Object[] items;
    private int numItems;

    // constructors go here

    public boolean add(Object item) {
        ...
}
```

- Each object of this class will represent an entire bag of items.
- The items themselves are stored in an array of type `Object`.
  - allows us to store *any* type of object in the bag, thanks to the power of polymorphism:

```
ArrayBag bag = new ArrayBag();
bag.add("hello");
bag.add(new Rectangle(20, 30));
```

Memory Management: Looking Under the Hood

- To understand how data structures are implemented, you need to understand how memory is managed.
- There are three main types of memory allocation in Java.
- They correspond to three different regions of memory.

Memory Management, Type I: Static Storage

- Static storage is used for *class variables*, which are declared *outside any method* using the keyword `static`:

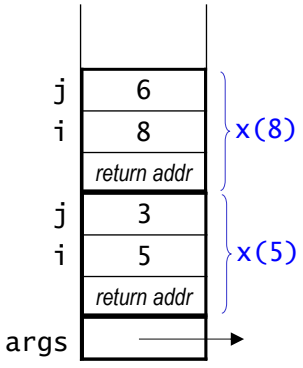
```
public class MyMethods {
    public static int numCompares;
    public static final double PI = 3.14159;
```

- There is only one copy of each class variable.
  - shared by all objects of the class
  - Java's version of a global variable
- The Java runtime allocates memory for class variables when the class is first encountered.
  - this memory stays fixed for the duration of the program

Memory Management, Type II: Stack Storage

- Method parameters and local variables are stored in a region of memory known as *the stack*.
- For each method call, a new *stack frame* is added to the top of the stack.

```
public class Foo {
  public static int x(int i) {
      int j = i - 2;
      if (i >= 6) {
          return i;   // return 8
      }
      return x(i + j);
  }
  public static void
    main(String[] args) {
      System.out.println(x(5));
  }
}
```



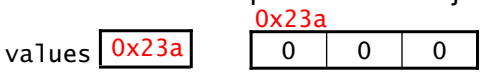
- When a method completes, its stack frame is removed.

Memory Management, Type III: Heap Storage

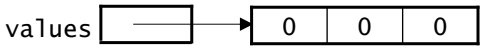
- Objects (including arrays) are stored in a region of memory known as *the heap*.
- Memory on the heap is allocated using the `new` operator:

```
int[] values = new int[3];
ArrayBag b = new ArrayBag();
```

- `new` returns the memory address of the start of the object.
- This memory address – which is referred to as a *reference* – is stored in the variable that represents the object:



- We will often use an arrow to represent a reference:



Heap Storage (cont.)

- An object remains on the heap until there are no remaining references to it.
- Unused objects are automatically reclaimed by a process known as *garbage collection*.
  - makes their memory available for other objects

Two Constructors for the `ArrayBag` Class

```
public class ArrayBag {
    private Object[] items;
    private int numItems;
    public static final int DEFAULT_MAX_SIZE = 50;

    public ArrayBag() {
        this.items = new Object[DEFAULT_MAX_SIZE];
        this.numItems = 0;
    }
    public ArrayBag(int maxSize) {
    ...
}
```

- A class can have multiple constructors.
  - the parameters must differ in some way
- The first one is useful for small bags.
  - creates an array with room for 50 items.
- The second one allows the client to specify the max # of items.

Two Constructors for the `ArrayBag` Class

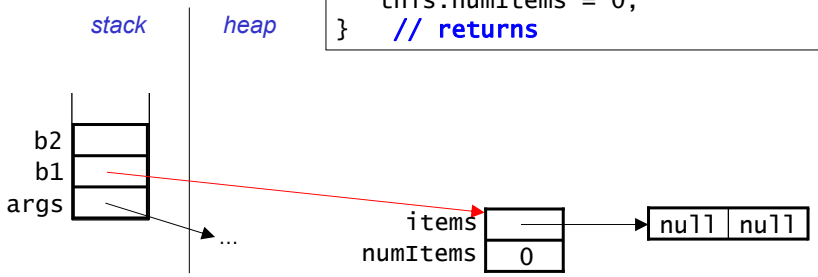
```
public class ArrayBag {
    private Object[] items;
    private int numItems;
    public static final int DEFAULT_MAX_SIZE = 50;

    public ArrayBag() {
        this.items = new Object[DEFAULT_MAX_SIZE];
        this.numItems = 0;
    }
    public ArrayBag(int maxSize) {
        if (maxSize <= 0) {
            throw new IllegalArgumentException(
              "maxSize must be > 0");
        }
        this.items = new Object[maxSize];
        this.numItems = 0;
    }
    ...
}
```

- If the user inputs an invalid `maxSize`, we throw an exception.

Example: Creating Two `ArrayBag` Objects

```
// client
public static void main(String[] args) {
    ArrayBag b1 = new ArrayBag(2);
    ArrayBag b2 = new ArrayBag(4);
    …
}
```

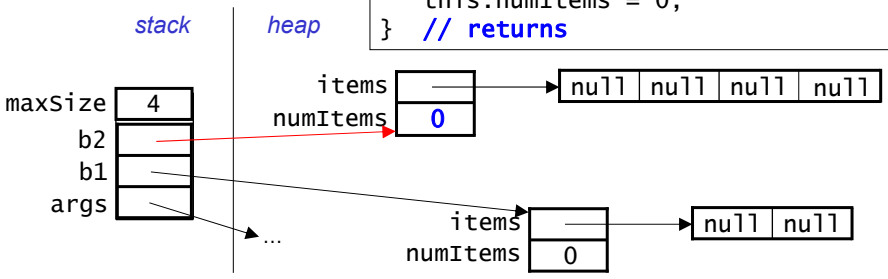
```
// constructor
public ArrayBag(int maxSize) {
   ...  // error-checking
   this.items = new Object[maxSize];
   this.numItems = 0;
}   // returns
```



Example: Creating Two `ArrayBag` Objects

```
// client
public static void main(String[] args) {
    ArrayBag b1 = new ArrayBag(2);
    ArrayBag b2 = new ArrayBag(4);
    …
}
```

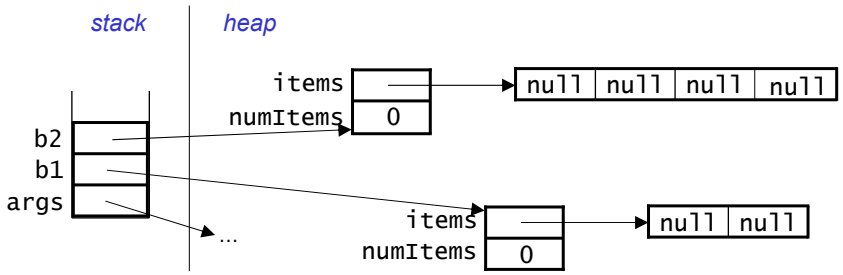
```
// constructor
public ArrayBag(int maxSize) {
   ...  // error-checking
   this.items = new Object[maxSize];
   this.numItems = 0;
}  // returns
```



Example: Creating Two `ArrayBag` Objects

```
// client
public static void main(String[] args) {
    ArrayBag b1 = new ArrayBag(2);
    ArrayBag b2 = new ArrayBag(4);
    …
}
```

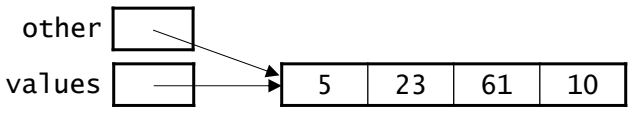
- After the objects have been created, here's what we have:



Copying References

- A variable that represents an array or object is known as a *reference variable*.
- Assigning the value of one reference variable to another reference variable copies the reference to the array or object. It does *not* copy the array or object itself.

```
int[] values = {5, 23, 61, 10};
int[] other = values;
```



- Given the lines above, what will the lines below output?

```
other[2] = 17;
System.out.println(values[2] + " " + other[2]);
```

Passing an Object/Array to a Method

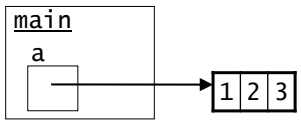
- When a method is passed an object or array as a parameter, the method gets a copy of the *reference* to the object or array, *not* a copy of the object or array itself.
- Thus, any changes that the method makes to the *internals* of the object/array will still be there when the method returns.
- Consider the following:

```
public static void main(String[] args) {
    int[] a = {1, 2, 3};
    triple(a);
    System.out.println(Arrays.toString(a));
}

public static void triple(int[] n) {
    for (int i = 0; i < n.length; i++) {
        n[i] = n[i] * 3;
    }
}
```

What is the output?

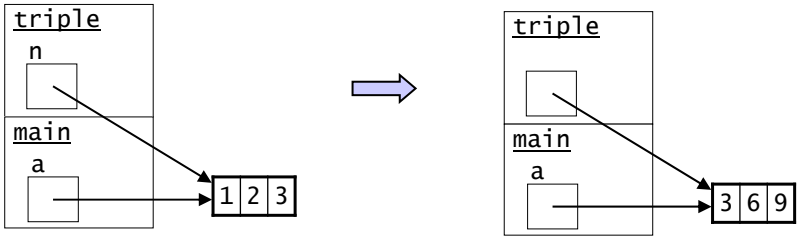
Passing an Object/Array to a Method (cont.)

before method call

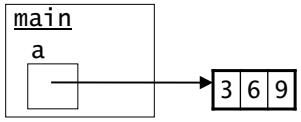



```
public static void main(String[] args) {
    int[] a = {1, 2, 3};
    triple(a);
    System.out.println(...);
}
```

during method call

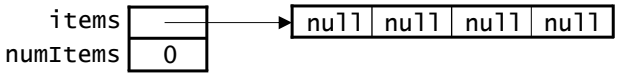


after method call

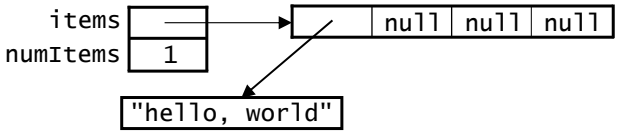


Adding Items to an `ArrayBag`

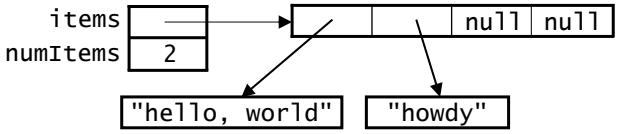
- We fill the array from left to right. Here's an empty bag:



- After adding the first item:

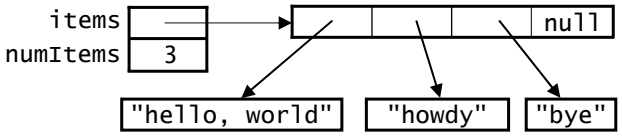


- After adding the second item:

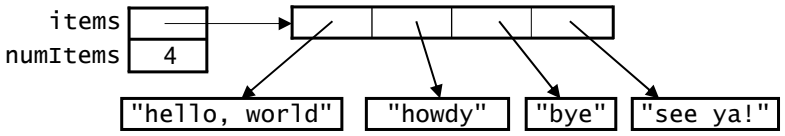


Adding Items (cont.)

- After adding the third item:



- After adding the fourth item:



- At this point, the `ArrayBag` is full!
  - it's non-trivial to "grow" an array, so we don't!
  - additional items cannot be added until one is removed

A Method for Adding an Item to a Bag

```
public class ArrayBag {
    private Object[] items;
    private int numItems;
    ...
    public boolean add(Object item) {
        if (item == null) {
            throw new IllegalArgumentException("no nulls");
        } else if (this.numItems == this.items.length) {
            return false;   // no more room!
        } else {
            this.items[this.numItems] = item;
            this.numItems++;
            return true;    // success!
        }
    }
    ...
}
```

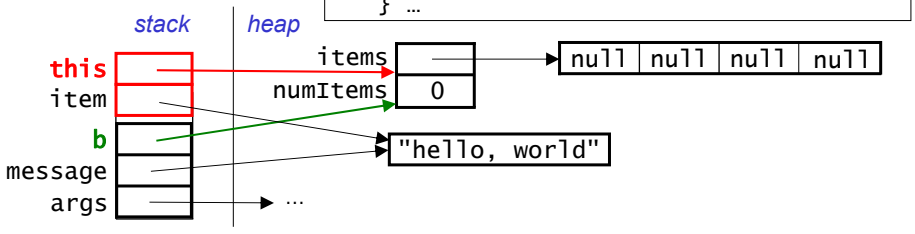
- takes an object of any type!
- returns a `boolean` to indicate if the operation succeeded

- Initially, `this.numItems` is 0, so the first item goes in position 0.
- We increase `this.numItems` because we now have 1 more item.
  - and so the *next* item added will go in the correct position!

Example: Adding an Item (cont.)

```
public static void main(String[] args) {
    String message = "hello, world";
    ArrayBag b = new ArrayBag(4);
    b.add(message);
    …
}
```

```
public boolean add(Object item) {
    …
    else {
        this.items[this.numItems] = item;
        this.numItems++;
        return true;
    } …
```

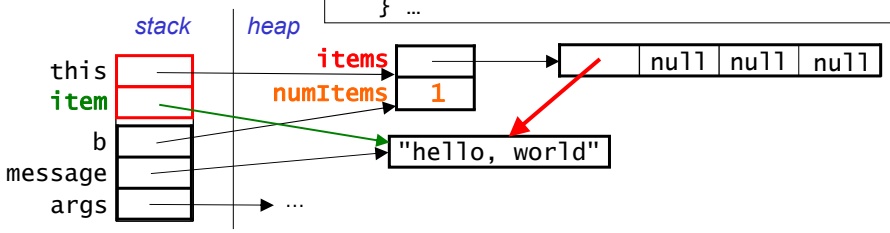


- add's stack frame includes:
  - `item`, which stores a copy of the reference passed as a param.
  - `this`, which stores a reference to the called `ArrayBag` object

Example: Adding an Item

```
public static void main(String[] args) {
    String message = "hello, world";
    ArrayBag b = new ArrayBag(4);
    b.add(message);
    …
}
```

```
public boolean add(Object item) {
    …
    else {
        this.items[this.numItems] = item;
        this.numItems++;
        return true;
    } …
```

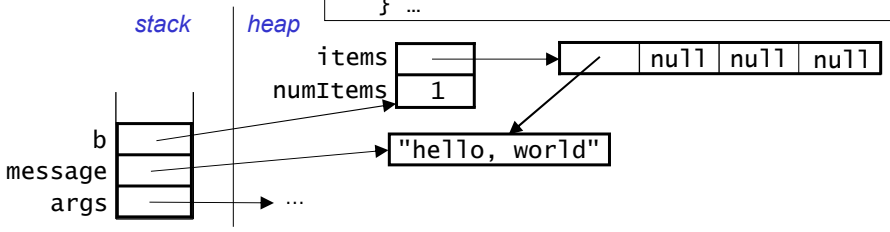


- The method modifies the `items` array and `numItems`.
  - note that the array holds a copy of the *reference* to the item, not a copy of the item itself.

Example: Adding an Item

```
public static void main(String[] args) {
    String message = "hello, world";
    ArrayBag b = new ArrayBag(4);
    b.add(message);
    …
}
```

```
public boolean add(Object item) {
    …
    else {
        this.items[this.numItems] = item;
        this.numItems++;
        return true;
    } …
```



- After the method call returns, add's stack frame is removed from the stack.

A Type Mismatch

- Here are the headers of two `ArrayBag` methods:

```
public boolean add(Object item)
public Object grab()
```

- Polymorphism allows us to pass `String` objects into `add()`:

```
ArrayBag stringBag = new ArrayBag();
stringBag.add("hello");
stringBag.add("world");
```

- However, this will not work:

```
String str = stringBag.grab();   // compiler error
```

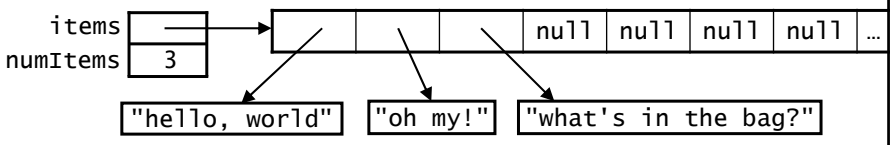
  - the return type of `grab()` is `Object`
  - `Object` isn't a subclass of `String`, so polymorphism doesn't help!

- Instead, we need to use a *type cast*:

```
String str = (String)stringBag.grab();
```

  - this cast doesn't actually change the value being assigned
  - it just reassures the compiler that the assignment is okay

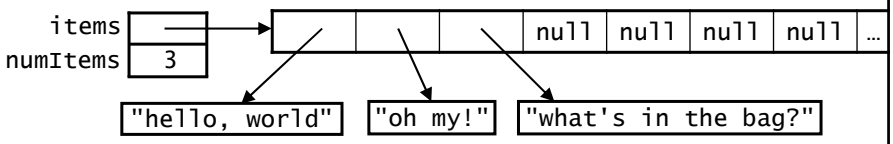
Extra Practice: Determining if a Bag Contains an Item



- Let's write the `ArrayBag` `contains()` method together.
  - should return `true` if an object equal to `item` is found, and `false` otherwise.

```
_________________ contains(_____________ item) {




}
```

Would this work instead?



- Let's write the `ArrayBag contains()` method together.
  - should return `true` if an object equal to `item` is found, and `false` otherwise.

```
public boolean contains(Object item) {
    for (int i = 0; i < this.items.length; i++) {
        if (this.items[i].equals(item)) { // not ==
            return true;
        }
    }
    return false;
}
```

Another Incorrect `contains()` Method

```
public boolean contains(Object item) {
    for (int i = 0; i < this.numItems; i++) {
        if (this.items[i].equals(item)) {
            return true;
        } else {
            return false;
        }
    }
    return false;
}
```

- What's the problem with this?

Recursion and Recursive Backtracking

Computer Science E-22
Harvard Extension School

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Iteration

- When we encounter a problem that requires repetition, we often use *iteration* – i.e., some type of loop.
- Sample problem: printing the series of integers from n1 to n2, where n1 <= n2.
  - example: `printSeries(5, 10)` should print the following:

    `5, 6, 7, 8, 9, 10`
- Here's an iterative solution to this problem:

```
public static void printSeries(int n1, int n2) {
    for (int i = n1; i < n2; i++) {
        System.out.print(i + ", ");
    }
    System.out.println(n2);
}
```

Recursion

- An alternative approach to problems that require repetition is to solve them using *recursion.*
- A recursive method is a method that calls itself.
- Applying this approach to the print-series problem gives:

```
public static void printSeries(int n1, int n2) {
    if (n1 == n2) {
        System.out.println(n2);
    } else {
        System.out.print(n1 + ", ");
        printSeries(n1 + 1, n2);
    }
}
```

Tracing a Recursive Method

```
public static void printSeries(int n1, int n2) {
    if (n1 == n2) {
        System.out.println(n2);
    } else {
        System.out.print(n1 + ", ");
        printSeries(n1 + 1, n2);
    }
}
```

- What happens when we execute `printSeries(5, 7)`?

```
printSeries(5, 7):
    System.out.print(5 + ", ");
    printSeries(6, 7):
        System.out.print(6 + ", ");
        printSeries(7, 7):
            System.out.print(7);
            return
        return
    return
```

Recursive Problem-Solving

- When we use recursion, we solve a problem by reducing it to a simpler problem of the same kind.
- We keep doing this until we reach a problem that is simple enough to be solved directly.
- This simplest problem is known as the *base case*.

```
public static void printSeries(int n1, int n2) {
    if (n1 == n2) {           // base case
        System.out.println(n2);
    } else {
        System.out.print(n1 + ", ");
        printSeries(n1 + 1, n2);
    }
}
```

- The base case stops the recursion, because it doesn't make another call to the method.

Recursive Problem-Solving (cont.)

- If the base case hasn't been reached, we execute the *recursive case.*

```
public static void printSeries(int n1, int n2) {
    if (n1 == n2) {           // base case
        System.out.println(n2);
    } else {                  // recursive case
        System.out.print(n1 + ", ");
        printSeries(n1 + 1, n2);
    }
}
```

- The recursive case:
  - reduces the overall problem to one or more simpler problems of the same kind
  - makes recursive calls to solve the simpler problems

Structure of a Recursive Method

```
recursiveMethod(parameters) {
    if (stopping condition) {
        // handle the base case
    } else {
        // recursive case:
        // possibly do something here

        recursiveMethod(modified parameters);

        // possibly do something here
    }
}
```

- There can be multiple base cases and recursive cases.
- When we make the recursive call, we typically use parameters that bring us closer to a base case.

Tracing a Recursive Method: Second Example

```
public static void mystery(int i) {
    if (i <= 0) {     // base case
        return;
    }
    // recursive case
    System.out.println(i);
    mystery(i - 1);
    System.out.println(i);
}
```

- What happens when we execute `mystery(2)`?

A Recursive Method That Returns a Value

- Simple example: summing the integers from 1 to n

```
public static int sum(int n) {
    if (n <= 0) {
        return 0;
    }
    int rest = sum(n - 1);
    return n + rest;
}
```

- Example of this approach to computing the sum:

```
sum(6)  =  6 + sum(5)
        =  6 + 5 + sum(4)
        =  …
```

Tracing a Recursive Method

```
public static int sum(int n) {
    if (n <= 0) {
        return 0;
    }
    int rest = sum(n - 1);
    return n + rest;
}
```

- What happens when we execute `int x = sum(3);` from inside the `main()` method?

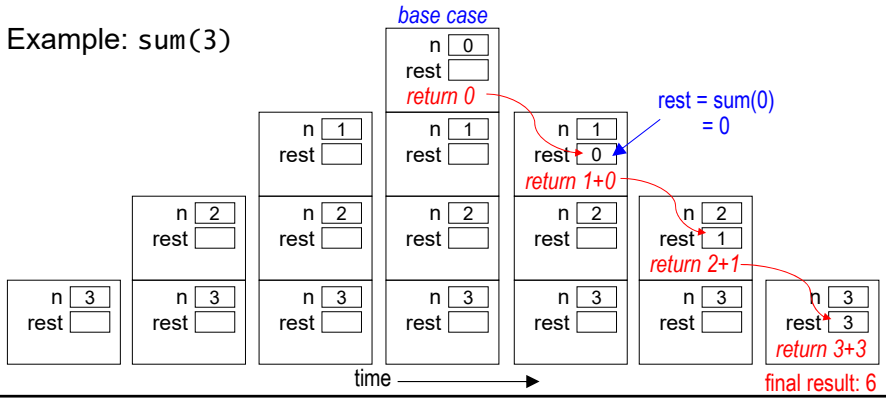
```
main() calls sum(3)
    sum(3) calls sum(2)
        sum(2) calls sum(1)
            sum(1) calls sum(0)
                sum(0) returns 0
            sum(1) returns 1 + 0 or 1
        sum(2) returns 2 + 1 or 3
    sum(3) returns 3 + 3 or 6
main() assigns 6 to x
```

Tracing a Recursive Method on the Stack

```
public static int sum(int n) {
    if (n <= 0) {
        return 0;
    }
    int rest = sum(n - 1);
    return n + rest;
}
```

The final result gets built up ***on the way back*** from the base case!

Example: `sum(3)`



Infinite Recursion

- We have to ensure that a recursive method will eventually reach a base case, regardless of the initial input.
- Otherwise, we can get *infinite recursion*.
  - produces *stack overflow* - there's no room for more frames on the stack!
- Example: here's a version of our `sum()` method that uses a different test for the base case:

```
public static int sum(int n) {
    if (n == 0) {
        return 0;
    }
    int rest = sum(n - 1);
    return n + rest;
}
```

  - what values of n would cause infinite recursion?

Designing a Recursive Method

1. Start by programming the base case(s).
   - *What instance(s) of this problem can I solve directly (without looking at anything smaller)?*
2. Find the recursive substructure.
   - *How could I use the solution to **any smaller version** of the problem to solve the overall problem?*
3. Solve the smaller problem using a recursive call!
   - store its result in a variable
4. Do your one step.
   - build your solution from the result of the recursive call
   - use *concrete cases* to figure out what you need to do

Processing a String Recursively

- A string is a recursive data structure. It is either:
  - empty ("")
  - a single character, followed by a string
- Thus, we can easily use recursion to process a string.
  - process one or two of the characters ourselves
  - make a recursive call to process the rest of the string
- Example: print a string vertically, one character per line:

```
public static void printVertical(String str) {
    if (str == null || str.equals("")) {
        return;
    }

    System.out.println(str.charAt(0)); // first char
    printVertical(str.substring(1));   // rest of string
}
```

Short-Circuited Evaluation

- The second operand of both the && and || operators will not be evaluated if the result can be determined on the basis of the first operand alone.

- `expr1 || expr2`
  if `expr1` evaluates to `true`, `expr2` is not evaluated, because we already know that `expr1 || expr2` is `true`
  - example from the last slide:

```
if (str == null || str.equals("")) {
    return;
}
// if str is null, we won't check for empty string.
// This prevents a null pointer exception!
```

- `expr1 && expr2`
  if `expr1` evaluates to _____, `expr2` is not evaluated, because we already know that `expr1 && expr2` is _____.

Counting Occurrences of a Character in a String

- `numOccur(c, s)` should return the number of times that the character `c` appears in the string `s`
  - `numOccur('n', "banana")` should return `2`
  - `numOccur('a', "banana")` should return `3`

- Take the approach outlined earlier:
  - base case: empty string (or null)
  - delegate `s.substring(1)` to the recursive call
  - we're responsible for handling `s.charAt(0)`

Recursive Counting

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        // do our one step!
        ...
```

Determining Our One Step

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        // do our one step!
```

- In our one step, we take care of `s.charAt(0)`.
  - we build the solution to the larger problem on the solution to the smaller problem (in this case, `rest`)
  - does what we do depend on the value of `s.charAt(0)`?

- ***Use concrete cases to figure out the logic!***

Consider this concrete case…

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        // do our one step!
        ...

numOccur('r', "recurse")
```

What value is eventually assigned to `rest`? (i.e., what does the recursive call return?)

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        // do our one step!
        ...

numOccur('r', "recurse")
```

```
numOccur('r', "recurse")
  c = 'r', s = "recurse"
  int rest = ???
```

Consider Concrete Cases

```
numOccur('r', "recurse")     # first char is a match
```

- what is its solution?
- what is the next smaller subproblem?
- what is the solution to that subproblem?
- how can we use the solution to the subproblem?
  What is our one step?

```
numOccur('a', "banana")     # first char is not a match
```

- what is its solution?
- what is the next smaller subproblem?
- what is the solution to that subproblem?
- how can we use the solution to the subproblem?
  What is our one step?

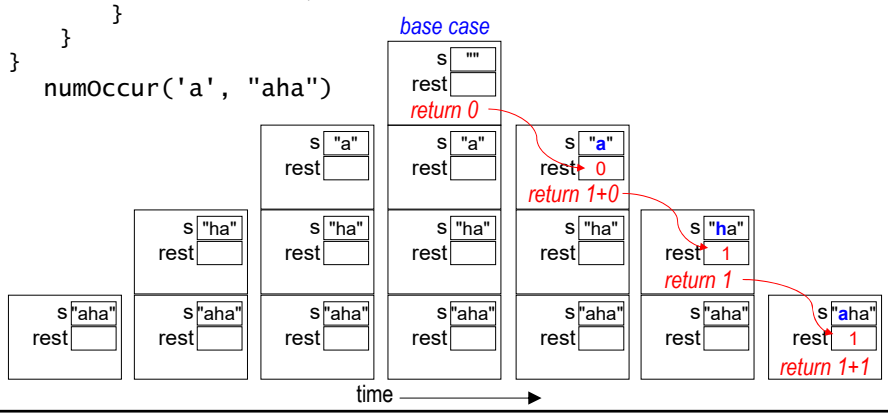
Now complete the method!

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        if (s.charAt(0) == c) {

            return ____________________;
        } else {

            return ____________________;
        }
    }
}
```

Tracing a Recursive Method on the Stack

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    } else {
        int rest = numOccur(c, s.substring(1));
        if (s.charAt(0) == c) {
            return 1 + rest;
        } else {
            return rest;
        }
    }
}
   numOccur('a', "aha")
```

The final result gets built up ***on the way back*** from the base case!



Common Mistake

- This version of the method does *not* work:

```
public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    }

    int count = 0;
    if (s.charAt(0) == c) {
        count++;
    }

    numOccur(c, s.substring(1));
    return count;
}
```

Another Faulty Approach

- Some people make `count` "global" to fix the prior version:

```
public static int count = 0;

public static int numOccur(char c, String s) {
    if (s == null || s.equals("")) {
        return 0;
    }

    if (s.charAt(0) == c) {
        count++;
    }

    numOccur(c, s.substring(1));
    return count;
}
```

- Not recommended, and not allowed on the problem sets!
- Problems with this approach?

Removing Vowels From a String

- `removeVowels(s)` - removes the vowels from the string `s`, returning its "vowel-less" version!

  `removeVowels("recursive")` should return `"rcrsv"`

  `removeVowels("vowel")` should return `"vwl"`

- Can we take the usual approach to recursive string processing?
  - base case: empty string
  - delegate `s.substring(1)` to the recursive call
  - we're responsible for handling `s.charAt(0)`

Applying the String-Processing Template

```
public static String removeVowels(String s) {
    if (s.equals("")) {    // base case

        return __________;
    } else {               // recursive case

        String rem_rest = ___________________;
        // do our one step!
    }
}
```

Consider Concrete Cases

```
removeVowels("after")     # first char is a vowel
```

- what is its solution?
- what is the next smaller subproblem?
- what is the solution to that subproblem?
- how can we use the solution to the subproblem?
  What is our one step?

```
removeVowels("recurse")   # first char is not a vowel
```

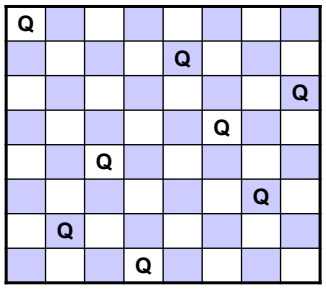
- what is its solution?
- what is the next smaller subproblem?
- what is the solution to that subproblem?
- how can we use the solution to the subproblem?
  What is our one step?

removeVowels()

```
public static String removeVowels(String s) {
    if (s.equals("")) {       // base case
        return "";
    } else {                  // recursive case
        String rem_rest = removeVowels(s.substring(1));
        if ("aeiou".indexOf(s.charAt(0)) != -1) {

            _____________________________________

        } else {

            _____________________________________

        }
    }
}
```

The n-Queens Problem

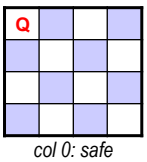
- **Goal:** to place n queens on an n x n chessboard so that no two queens occupy:
  - the same row
  - the same column
  - the same diagonal.
- Sample solution for n = 8:



- This problem can be solved using a technique called *recursive backtracking*.

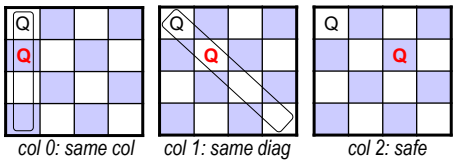
Recursive Strategy for n-Queens

- `findSolution(row)` – to place a queen in the specified `row`:
  - try one column at a time, looking for a "safe" one
  - if we find one: – place the queen there
    – *make a recursive call* to go to the next row
  - if we can't find one: – *backtrack* by returning from the call
    – try to find another safe column in the previous row
- Example:
  - row 0:



col 0: safe

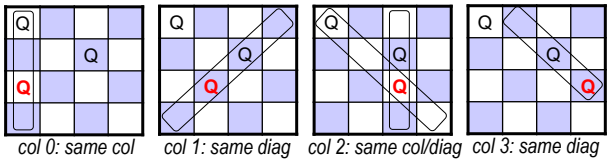
  - row 1:



col 0: same col *col 1: same diag* *col 2: safe*

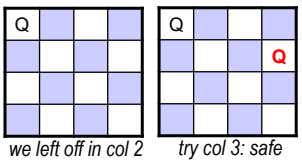
4-Queens Example (cont.)

- row 2:



col 0: same col *col 1: same diag* *col 2: same col/diag* *col 3: same diag*

- We've run out of columns in row 2!
- *Backtrack* to row 1 by returning from the recursive call.
  - pick up where we left off
  - we had already tried columns 0-2, so now we try column 3:

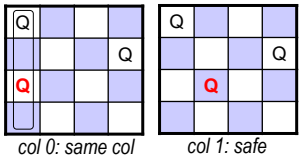


we left off in col 2 *try col 3: safe*

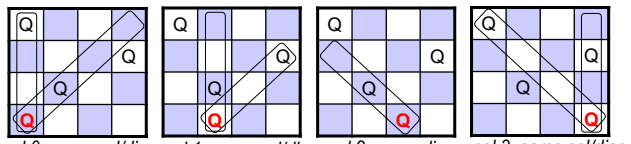
- Continue the recursion as before.

4-Queens Example (cont.)

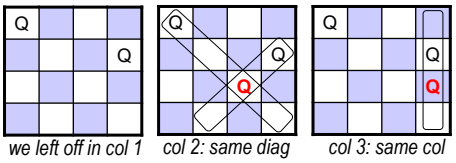
- row 2:



- row 3:



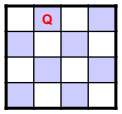
- Backtrack to row 2:



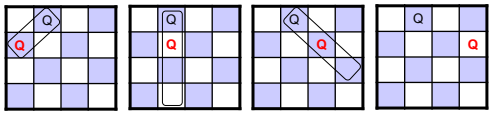
- Backtrack to row 1. No columns left, so backtrack to row 0!

4-Queens Example (cont.)

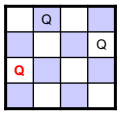
- row 0:



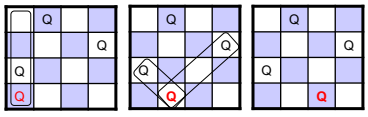
- row 1:



- row 2:



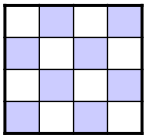
- row 3:



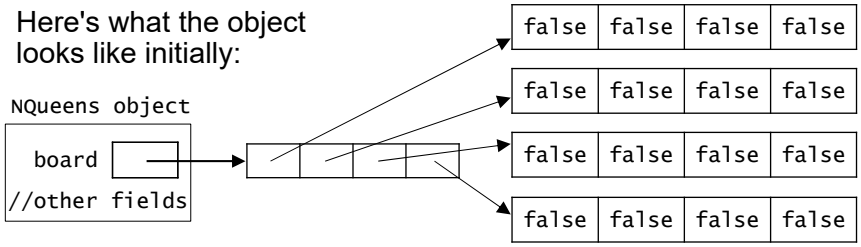
A solution!

A Blueprint Class for an N-Queens Solver

```
public class NQueens {
    private boolean[][] board;  // state of the chessboard
    // other fields go here...

    public NQueens(int n) {
        this.board = new boolean[n][n];
        // initialize other fields here...
    }

    ...
```

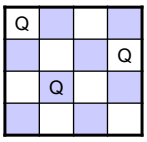


- Here's what the object looks like initially:

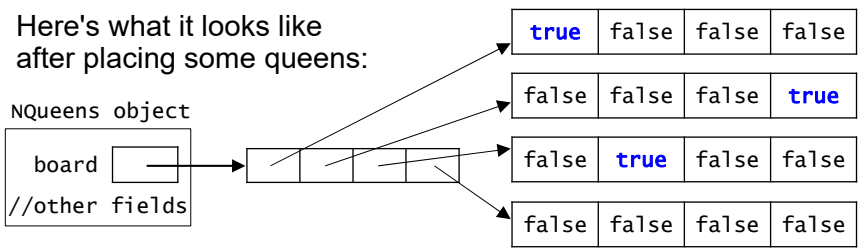


A Blueprint Class for an N-Queens Solver

```
public class NQueens {
    private boolean[][] board;  // state of the chessboard
    // other fields go here...

    public NQueens(int n) {
        this.board = new boolean[n][n];
        // initialize other fields here...
    }

    private void placeQueen(int row, int col) {
        this.board[row][col] = true;
        // modify other fields here...
    }
```



- Here's what it looks like after placing some queens:



A Blueprint Class for an N-Queens Solver

```
public class NQueens {
    private boolean[][] board;  // state of the chessboard
    // other fields go here...

    public NQueens(int n) {
        this.board = new boolean[n][n];
        // initialize other fields here...
    }

    private void placeQueen(int row, int col) {
        this.board[row][col] = true;
        // modify other fields here...
    }

    private void removeQueen(int row, int col){
        this.board[row][col] = false;
        // modify other fields here...
    }

    private boolean isSafe(int row, int col) {
        // returns true if [row][col] is "safe", else false
    }

    private boolean findSolution(int row) {
        // see next slide!
    ...
```

private helper methods that will only be called by code within the class.

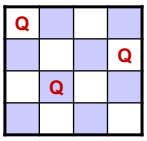
Making them private means we don't need to do error-checking!

Recursive-Backtracking Method

```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) {
                return true;
            }
            this.removeQueen(row, col);
        }
    }
    return false;
}
```

- takes the index of a row (initially 0)
- uses a loop to consider all possible columns in that row
- makes a recursive call to move onto the next row
- returns `true` if a solution has been found; `false` otherwise

Tracing findSolution()




```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        // code to process a solution goes here...
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) {
                return true;
            }
            this.removeQueen(row, col);
        }
    }
    return false;
}
```

Note: row++ will *not* work here!

We can pick up where we left off, because row *and* col *are stored in the stack frame!*

backtrack!

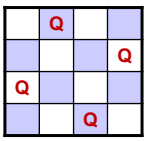
backtrack!

row: 0 col: 0	row: 1 col: 0,1,2	row: 2 col:0,1,2,3,4 return false	row: 1 col: 0,1,2	row: 0 col: 0	row: 1 col: 0,1,2,3	row: 0 col: 0	row: 2 col: 0,1	row: 1 col: 0,1,2,3	row: 0 col: 0	row: 3 col:**0,1,2,3,4** return false	row: 2 col: 0,1	row: 1 col: 0,1,2,3	row: 0 col: 0
					row: 1 col: 0,1,2,***3***								

…

time →

Once we place a queen in the last row…



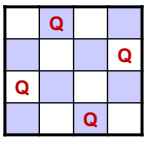

```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) {
                return true;
            }
            this.removeQueen(row, col);
        }
    }
    return false;
}
```

…

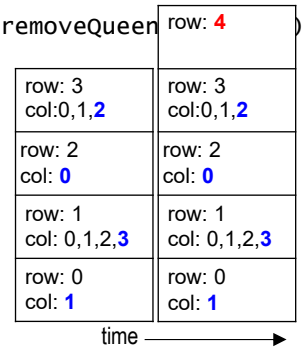
row: 3 col:0,1,**2**
row: 2 col: **0**
row: 1 col: 0,1,2,**3**
row: 0 col: **1**

time →

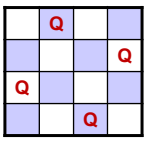
…we make one more recursive call…



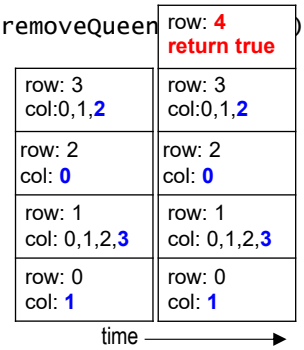

```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) {
                return true;
            }
            this.removeQueen(row, col);
        }
    }
    return false;
}
```



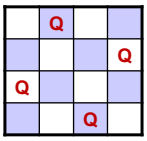
…and hit the base case!



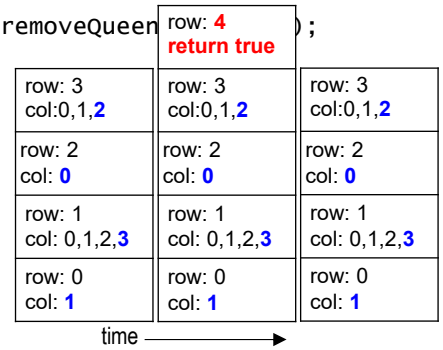

```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) {
                return true;
            }
            this.removeQueen(row, col);
        }
    }
    return false;
}
```



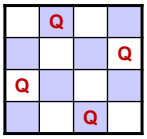
`true` is sent back…



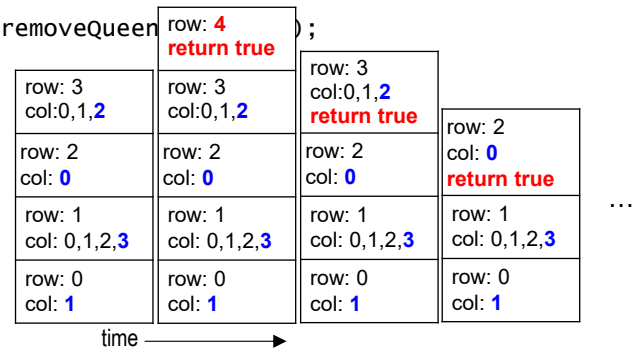

```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) { // if (true)
                return true;
            }
            this.removeQueen                    );
        }
    }
    return false;
}
```



…and all the earlier calls also return `true`!




```
private boolean findSolution(int row) {
    if (row == this.board.length) {
        this.displayBoard();
        return true;
    }
    for (int col = 0; col < this.board.length; col++) {
        if (this.isSafe(row, col)) {
            this.placeQueen(row, col);
            if (this.findSolution(row + 1)) { // if (true)
                return true;
            }
            this.removeQueen                    );
        }
    }
    return false;
}
```



Using a "Wrapper" Method

- The key recursive method is private:

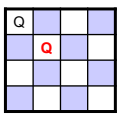
```
private boolean findSolution(int row) {
    ...
}
```

- We use a separate, public *"wrapper" method* to start the recursion:

```
public boolean findSolution() {
    return this.findSolution(0);
}
```

  - an example of overloading – two methods with the same name, but different parameters
  - this method takes no parameters
  - it makes the initial call to the recursive method and returns whatever that call returns
  - it allows us to ensure that the correct initial value is passed into the recursive method

Recursive Backtracking in General

- Useful for *constraint satisfaction problems*
  - involve assigning values to variables according to a set of constraints
  - n-Queens: variables = Queen's position in each row
    constraints = no two queens in same row/col/diag
  - many others: factory scheduling, room scheduling, etc.
- Backtracking greatly reduces the number of possible solutions that we consider.
  - ex: 
    - there are 16 possible solutions that begin with queens in these two positions
    - backtracking doesn't consider any of them!
- Recursion makes it easy to handle an arbitrary problem size.
  - stores the state of each variable in a separate stack frame

Template for Recursive Backtracking

```
// n is the number of the variable that the current
// call of the method is responsible for
boolean findSolution(int n, possibly other params) {
    if (found a solution) {
        this.displaySolution();
        return true;
    }
    // loop over possible values for the nth variable
    for (val = first to last) {
        if (this.isValid(val, n)) {
            this.applyValue(val, n);
            if (this.findSolution(n + 1, other params)) {
                return true;
            }
            this.removeValue(val, n);
        }
    }
    return false;     // backtrack!
}
```

Note: n++ will *not* work here!

Template for Finding Multiple Solutions

(up to some target number of solutions)

```
boolean findSolutions(int n, possibly other params) {
    if (found a solution) {
        this.displaySolution();
        this.solutionsFound++;
        return (this.solutionsFound >= this.target);
    }

    // loop over possible values for the nth variable
    for (val = first to last) {
        if (isValid(val, n)) {
            this.applyValue(val, n);
            if (this.findSolutions(n + 1, other params)) {
                return true;
            }
            this.removeValue(val, n);
        }
    }

    return false;
}
```

Data Structures for n-Queens

- Three key operations:
  - `isSafe(row, col)`: check to see if a position is safe
  - `placeQueen(row, col)`
  - `removeQueen(row, col)`
- In theory, our 2-D array of booleans would be sufficient:

```
public class NQueens {
    private boolean[][] board;
```

- It's easy to place or remove a queen:

```
    private void placeQueen(int row, int col) {
        this.board[row][col] = true;
    }
    private void removeQueen(int row, int col) {
        this.board[row][col] = false;
    }
…
```

- Problem: `isSafe()` takes a lot of steps. What matters more?

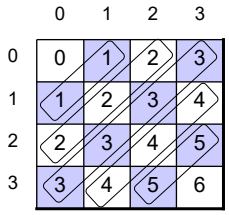
Additional Data Structures for n-Queens

- To facilitate `isSafe()`, add three arrays of booleans:

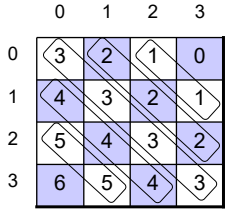
```
private boolean[] colEmpty;
private boolean[] upDiagEmpty;
private boolean[] downDiagEmpty;
```

- An entry in one of these arrays is:
  - `true` if there are no queens in the column or diagonal
  - `false` otherwise
- Numbering diagonals to get the indices into the arrays:

upDiag = row + col



downDiag = (boardSize – 1) + row – col



Using the Additional Arrays

- Placing and removing a queen now involve updating four arrays instead of just one. For example:

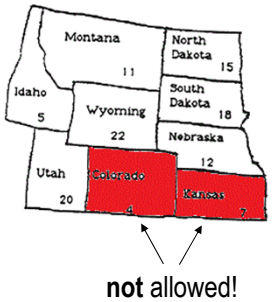
```
private void placeQueen(int row, int col) {
    this.board[row][col] = true;
    this.colEmpty[col] = false;
    this.upDiagEmpty[row + col] = false;
    this.downDiagEmpty[
         (this.board.length - 1) + row - col] = false;
}
```

- However, checking if a square is safe is now more efficient:

```
private boolean isSafe(int row, int col) {
    return (this.colEmpty[col]
      && this.upDiagEmpty[row + col]
      && this.downDiagEmpty[
              (this.board.length - 1) + row - col]);
}
```

Recursive Backtracking II: Map Coloring

- We want to color a map using **only four colors**.
- Bordering states or countries ***cannot*** have the same color.
  - example:



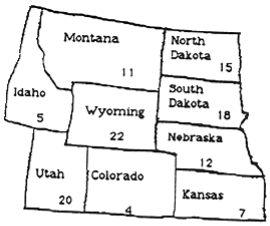
Applying the Template to Map Coloring

```
boolean findSolution(n, perhaps other params) {
    if (found a solution) {
        this.displaySolution();
        return true;
    }
    for (val = first to last) {
        if (this.isValid(val, n)) {
            this.applyValue(val, n);
            if (this.findSolution(n + 1, other params)) {
                return true;
            }
            this.removeValue(val, n);
        }
    }
    return false;
}
```

template element	*meaning in map coloring*
n	
found a solution	
val	
isValid(val, n)	
applyValue(val, n)	
removeValue(val, n)	

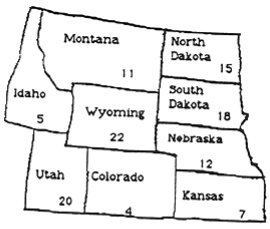
Map Coloring Example

consider the states in alphabetical order. colors = { red, yellow, green, blue }.



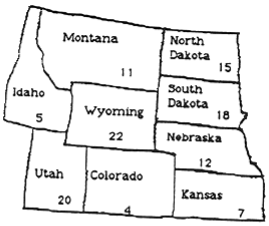
We color Colorado through Utah without a problem.

Colorado:
Idaho:
Kansas:
Montana:
Nebraska:
North Dakota:
South Dakota:
Utah:



No color works for Wyoming, so we backtrack…

Map Coloring Example (cont.)



Now we can complete the coloring:

Recursion vs. Iteration

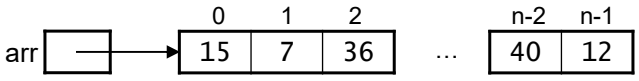
- Some problems are much easier to solve using recursion.
- Other problems are just as easy to solve using iteration.
- Recursion is a bit more costly because of the overhead involved in invoking a method.
  - also: in some cases, there may not be room on the stack
- Rule of thumb:
  - if it's easier to formulate a solution recursively, use recursion, unless the cost of doing so is too high
  - otherwise, use iteration

A First Look at Sorting and Algorithm Analysis

Computer Science E-22
Harvard Extension School

David G. Sullivan, Ph.D.

Sorting an Array of Integers



- Ground rules:
  - sort the values in increasing order
  - sort "in place," using only a small amount of additional storage
- Terminology:
  - position: one of the memory locations in the array
  - element: one of the data items stored in the array
  - element i: the element at position i
- Goal: minimize the number of **comparisons *C*** and the number of **moves *M*** needed to sort the array.
  - move = copying an element from one position to another
    example: `arr[3] = arr[5];`

Defining a Class for our Sort Methods

```
public class Sort {
    public static void bubbleSort(int[] arr) {
        ...
    }
    public static void insertionSort(int[] arr) {
        ...
    }
    ...
}
```

- Our `Sort` class is simply a collection of methods like Java's built-in `Math` class.
- Because we never create `Sort` objects, all of the methods in the class must be *static*.
  - outside the class, we invoke them using the class name: e.g., `Sort.bubbleSort(arr)`

Defining a Swap Method

- It would be helpful to have a method that swaps two elements of the array.
- Why won't the following work?

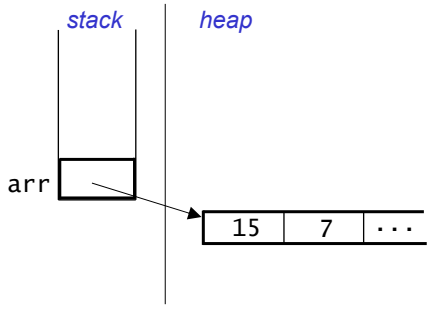
```
private static void swap(int a, int b) {
    int temp = a;
    a = b;
    b = temp;
}
```

An Incorrect Swap Method

```
private static void swap(int a, int b) {
    int temp = a;
    a = b;
    b = temp;
}
```

- Trace through the following lines to see the problem:

```
int[] arr = {15, 7, …};
swap(arr[0], arr[1]);
```



A Correct Swap Method

- This method works:

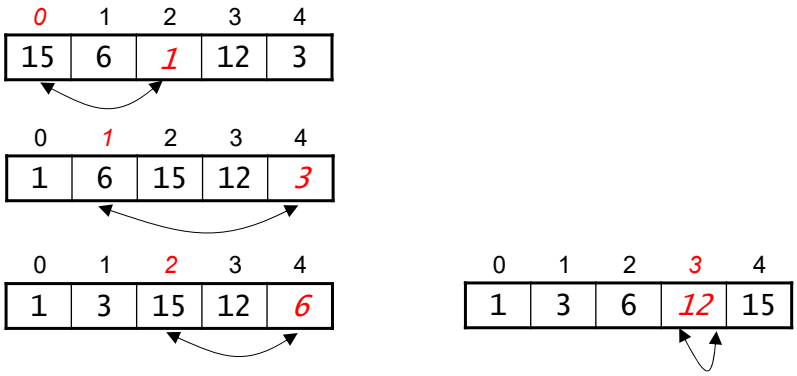
```
private static void swap(int[] arr, int a, int b) {
    int temp = arr[a];
    arr[a] = arr[b];
    arr[b] = temp;
}
```

- Trace through the following with a memory diagram to convince yourself that it works:

```
int[] arr = {15, 7, …};
swap(arr, 0, 1);
```

Selection Sort

- Basic idea:
  - consider the positions in the array from left to right
  - for each position, find the element that belongs there and swap it with the element that's currently there
- Example:

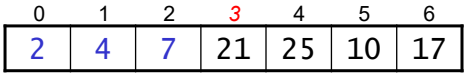


Why don't we need to consider position 4?

Selecting an Element

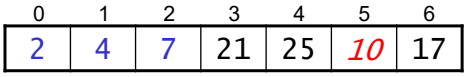
- When we consider position `i`, the elements in positions 0 through `i - 1` are already in their final positions.

example for `i = 3`:

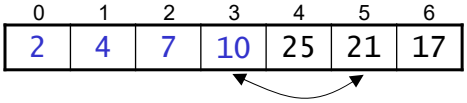


- To select an element for position `i`:
  - consider elements `i`, `i+1`, `i+2`, …, `arr.length - 1`, and keep track of `indexMin`, the index of the smallest element seen thus far

`indexMin: 3, 5`



  - when we finish this pass, `indexMin` is the index of the element that belongs in position `i`.
  - swap `arr[i]` and `arr[indexMin]`:



Implementation of Selection Sort

- Use a helper method to find the index of the smallest element:

```
private static int indexSmallest(int[] arr, int start) {
    int indexMin = start;

    for (int i = start + 1; i < arr.length; i++) {
        if (arr[i] < arr[indexMin]) {
            indexMin = i;
        }
    }

    return indexMin;
}
```

- The actual sort method is very simple:

```
public static void selectionSort(int[] arr) {
    for (int i = 0; i < arr.length - 1; i++) {
        int j = indexSmallest(arr, i);
        swap(arr, i, j);
    }
}
```

Time Analysis

- Some algorithms are much more efficient than others.
- The *time efficiency* or *time complexity* of an algorithm is some measure of the number of operations that it performs.
  - for sorting, we'll focus on comparisons and moves
- We want to characterize how the number of operations depends on the size, `n`, of the input to the algorithm.
  - for sorting, `n` is the length of the array
  - how does the number of operations grow as `n` grows?
- We'll express the number of operations as functions of `n`
  - `C(n)` = number of comparisons for an array of length `n`
  - `M(n)` = number of moves for an array of length `n`

Counting Comparisons by Selection Sort

```
private static int indexSmallest(int[] arr, int start){
    int indexMin = start;

    for (int i = start + 1; i < arr.length; i++) {
        if (arr[i] < arr[indexMin]) {
            indexMin = i;
        }
    }

    return indexMin;
}
public static void selectionSort(int[] arr) {
    for (int i = 0; i < arr.length - 1; i++) {
        int j = indexSmallest(arr, i);
        swap(arr, i, j);
    }
}
```

- To sort n elements, selection sort performs n – 1 passes:

  on 1st pass, it performs ____ comparisons to find `indexSmallest`
  on 2nd pass, it performs ____ comparisons
  …
  on the (n–1)st pass, it performs 1 comparison

- Adding them up: $C(n) = 1 + 2 + \ldots + (n - 2) + (n - 1)$

Counting Comparisons by Selection Sort (cont.)

- The resulting formula for $C(n)$ is the sum of an arithmetic sequence:

$$C(n) = 1 + 2 + \ldots + (n - 2) + (n - 1) = \sum_{i=1}^{n-1} i$$

- Formula for the sum of this type of arithmetic sequence:

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

- Thus, we can simplify our expression for C(n) as follows:

$$\begin{aligned} C(n) &= \sum_{i=1}^{n-1} i \\ &= \frac{(n-1)((n-1)+1)}{2} \\ &= \frac{(n-1)n}{2} \end{aligned}$$

$$C(n) = n^2/2 - n/2$$

Focusing on the Largest Term

- When n is large, mathematical expressions of n are dominated by their "largest" term — i.e., the term that grows fastest as a function of n.
  - example:

n	$n^2/2$	$n/2$	$n^2/2 - n/2$
10	50	5	45
100	5000	50	4950
10000	50,000,000	5000	49,995,000

- In characterizing the time complexity of an algorithm, we'll focus on the largest term in its operation-count expression.
  - for selection sort, $C(n) = n^2/2 - n/2 \approx n^2/2$
- In addition, we'll typically ignore the coefficient of the largest term (e.g., $n^2/2 \rightarrow n^2$).

Big-*O* Notation

- We specify the largest term using big-*O* notation.
  - e.g., we say that $C(n) = n^2/2 - n/2$ is $O(n^2)$
- Common classes of algorithms:

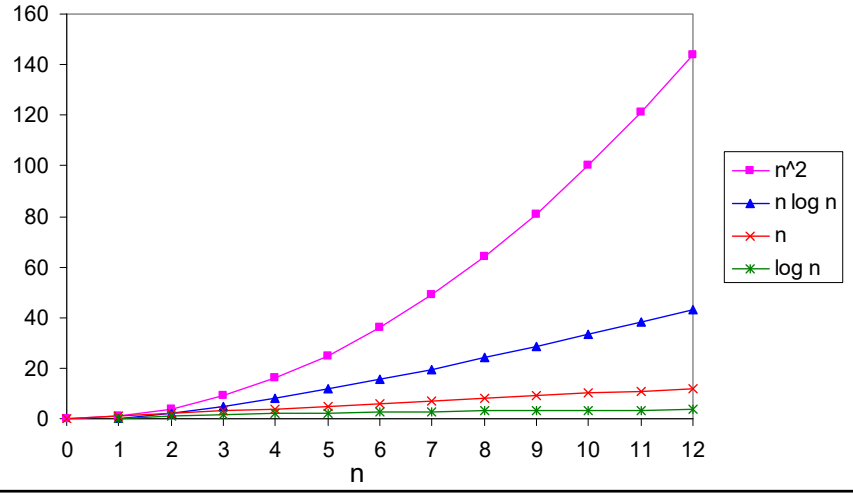
name	example expressions	big-O notation
constant time	$1, 7, 10$	$O(1)$
logarithmic time	$3\log_{10}n, \log_2 n + 5$	$O(\log n)$
linear time	$5n, 10n - 2\log_2 n$	$O(n)$
nlogn time	$4n\log_2 n, n\log_2 n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n, n^2 - 1$	$O(n^2)$
exponential time	$2^n, 5e^n + 2n^2$	$O(c^n)$

slower ↓

- For large inputs, efficiency matters more than CPU speed.
  - e.g., an $O(\log n)$ algorithm on a slow machine will outperform an $O(n)$ algorithm on a fast machine

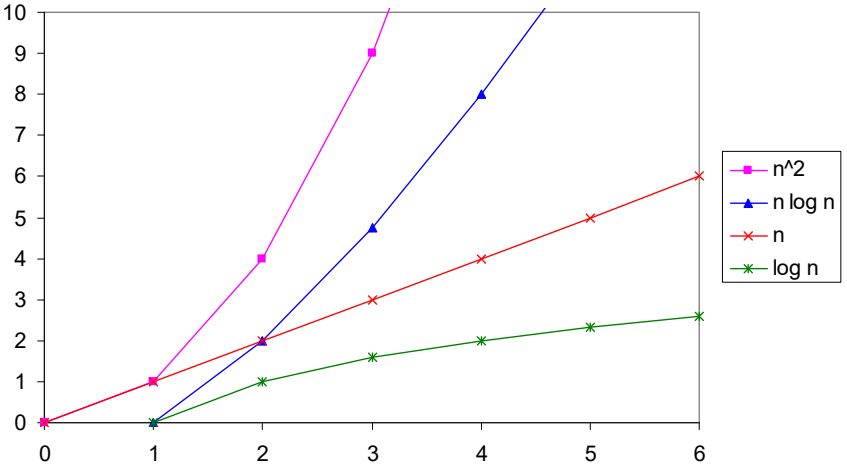
Ordering of Functions

- We can see below that: n^2 grows faster than $n \log_2 n$
  $n \log_2 n$ grows faster than n
  n grows faster than $\log_2 n$



Ordering of Functions (cont.)

- Zooming in, we see that: $n^2 >= n$ for all $n >= 1$
  $n \log_2 n >= n$ for all $n >= 2$
  $n > \log_2 n$ for all $n >= 1$

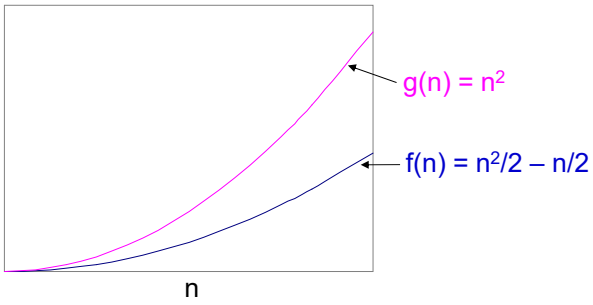


Big-*O* Time Analysis of Selection Sort

- Comparisons: we showed that $C(n) = n^2/2 - n/2$
  - selection sort performs $O(n^2)$ comparisons
- Moves: after each of the $n-1$ passes, the algorithm does one swap.
  - $n-1$ swaps, 3 moves per swap
  - $M(n) = 3(n-1) = 3n-3$
  - selection sort performs $O(n)$ moves.
- Running time (i.e., total operations): ?

Mathematical Definition of Big-*O* Notation

- $f(n) = O(g(n))$ if there exist positive constants c and n_0 such that $f(n) <= cg(n)$ for all $n >= n_0$
- Example: $f(n) = n^2/2 - n/2$ is $O(n^2)$, because
  $n^2/2 - n/2 <= n^2$ for all $n >= 0$.
  $c = 1$ $n_0 = 0$



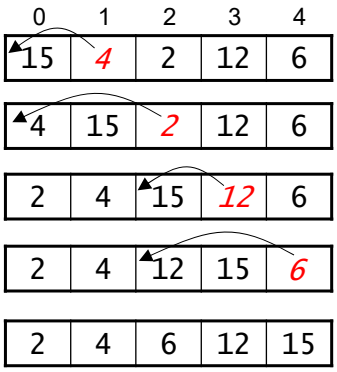
- Big-*O* notation specifies an *upper bound* on a function f(n) as n grows large.

Big-*O* Notation and Tight Bounds

- Strictly speaking, big-O notation provides an upper bound, *not* a tight bound (upper and lower).
- Example:
  - $3n - 3$ is $O(n^2)$ because $3n - 3 <= n^2$ for all $n >= 1$
  - $3n - 3$ is also $O(2^n)$ because $3n - 3 <= 2^n$ for all $n >= 1$
- However, it is common to use big-O notation to characterize a function as closely as possible – as if it specified a tight bound.
  - for our example, we would say that $3n - 3$ is $O(n)$
  - this is how you should use big-O in this class!

Insertion Sort

- Basic idea:
  - going from left to right, "insert" each element into its proper place with respect to the elements to its left
  - "slide over" other elements to make room
- Example:



Comparing Selection and Insertion Strategies

- In selection sort, we start with the *positions* in the array and *select* the correct elements to fill them.
- In insertion sort, we start with the *elements* and determine where to *insert* them in the array.
- Here's an example that illustrates the difference:

0	1	2	3	4	5	6
18	12	15	9	25	2	17

- Sorting by selection:
  - consider position 0: find the element (2) that belongs there
  - consider position 1: find the element (9) that belongs there
  - …
- Sorting by insertion:
  - consider the 12: determine where to insert it
  - consider the 15; determine where to insert it
  - …

Inserting an Element

- When we consider element `i`, elements 0 through `i - 1` are already sorted with respect to each other.

example for `i = 3`:

0	1	2	3	4
6	14	19	9	…

- To insert element `i`:
  - make a copy of element `i`, storing it in the variable `toInsert`:

`toInsert`	9

0	1	2	3
6	14	19	9

  - consider elements `i-1`, `i-2`, …
    - if an element `> toInsert`, slide it over to the right
    - stop at the first element `<= toInsert`

`toInsert`	9

0	1	2	3
6		14	19

  - copy `toInsert` into the resulting "hole":

0	1	2	3
6	9	14	19

Insertion Sort Example (done together)

description of steps

12	5	2	13	18	4

Implementation of Insertion Sort

```
public class Sort {
  ...
  public static void insertionSort(int[] arr) {
      for (int i = 1; i < arr.length; i++) {
          if (arr[i] < arr[i-1]) {
              int toInsert = arr[i];

              int j = i;
              do {
                  arr[j] = arr[j-1];
                  j = j - 1;
              } while (j > 0  &&  toInsert < arr[j-1]);

              arr[j] = toInsert;
          }
      }
  }
}
```

Time Analysis of Insertion Sort

- The number of operations depends on the contents of the array.
- *best case:* array is sorted
  - each element is only compared to the element to its left
  - we never execute the do-while loop!
  - `C(n)` =_____, `M(n)` = _____, running time = _____

also true if array is almost sorted

- *worst case:* array is in reverse order
  - each element is compared to *all* of the elements to its left:
    `arr[1]` is compared to `1` element (`arr[0]`)
    `arr[2]` is compared to `2` elements (`arr[0]` and `arr[1]`)
    …
    `arr[n-1]` is compared to `n-1` elements
  - `C(n)` = 1 + 2 + … + (n – 1) = _____
  - similarly, `M(n)` = _____, running time = _____
- *average case:* elements are randomly arranged
  - on average, each element is compared to *half* of the elements to its left
  - still get `C(n)` = `M(n)` = _____, running time = _____

Shell Sort

- Developed by Donald Shell
- Improves on insertion sort
  - takes advantage of the fact that it's fast for almost-sorted arrays
  - eliminates a key disadvantage: an element may need to move many times to get to where it belongs.
- Example: if the largest element starts out at the beginning of the array, it moves one place to the right on *every* insertion!

0	1	2	3	4	5	…	1000
999	42	56	30	18	23	…	11

- Shell sort uses larger moves that allow elements to quickly get close to where they belong in the sorted array.

Sorting Subarrays

- Basic idea:
  - use insertion sort on subarrays that contain elements separated by some increment `incr`
    - increments allow the data items to make larger "jumps"
  - repeat using a decreasing sequence of increments
- Example for an initial increment of 3:

0	1	2	3	4	5	6	7
36	18	10	27	3	20	9	8

  - three subarrays:
    1) elements 0, 3, 6 2) elements 1, 4, 7 3) elements 2 and 5
- Sort the subarrays using insertion sort to get the following:

0	1	2	3	4	5	6	7
9	3	10	27	8	20	36	18

- Next, we complete the process using an increment of 1.

Shell Sort: A Single Pass

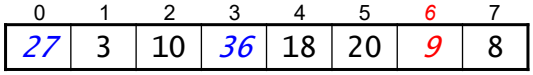
- We *don't* actually consider the subarrays one at a time.
- For each element from position `incr` to the end of the array, we insert the element into its proper place with respect to the elements *from its subarray* that come before it.
- The same example (`incr` = 3):

0	1	2	3	4	5	6	7
36	18	10	*27*	3	20	9	8
27	*18*	10	36	*3*	20	9	8
27	3	*10*	36	18	*20*	9	8
27	3	10	*36*	18	20	*9*	8
9	*3*	10	27	*18*	20	36	*8*
9	3	10	27	8	20	36	18

Inserting an Element in a Subarray

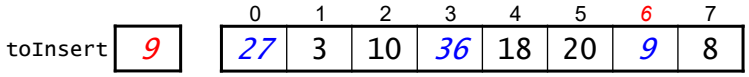
- When we consider element `i`, the other elements in its subarray are already sorted with respect to each other.

example for `i = 6`:
`(incr = 3)`

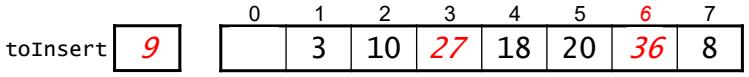


the other element's in 9's subarray (the 27 and 36)
are already sorted with respect to each other

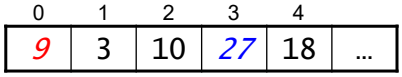
- To insert element `i`:
  - make a copy of element `i`, storing it in the variable `toInsert`:



  - consider elements `i-incr`, `i-(2*incr)`, `i-(3*incr)`,…
    - if an element `>` `toInsert`, slide it right *within the subarray*
    - stop at the first element `<=` `toInsert`



  - copy `toInsert` into the "hole":



The Sequence of Increments

- Different sequences of decreasing increments can be used.
- Our version uses values that are one less than a power of two.
  - $2^k - 1$ for some k
  - … 63, 31, 15, 7, 3, 1
  - can get to the next lower increment using integer division:

```
incr = incr/2;
```

- Should avoid numbers that are multiples of each other.
  - otherwise, elements that are sorted with respect to each other in one pass are grouped together again in subsequent passes
    - repeat comparisons unnecessarily
    - get fewer of the large jumps that speed up later passes
  - example of a bad sequence: 64, 32, 16, 8, 4, 2, 1
    - what happens if the largest values are all in odd positions?

Implementation of Shell Sort

```
public static void shellSort(int[] arr) {
    int incr = 1;
    while (2 * incr <= arr.length) {
        incr = 2 * incr;
    }
    incr = incr - 1;

    while (incr >= 1) {
        for (int i = incr; i < arr.length; i++) {
            if (arr[i] < arr[i-incr]) {
                int toInsert = arr[i];

                int j = i;
                do {
                     arr[j] = arr[j-incr];
                     j = j - incr;
                } while (j > incr-1 &&
                   toInsert < arr[j-incr]);

                arr[j] = toInsert;
            }
        }
        incr = incr/2;
    }
}
```

(If you replace `incr` with `1` in the for-loop, you get the code for insertion sort.)

Time Analysis of Shell Sort

- Difficult to analyze precisely
  - typically use experiments to measure its efficiency
- With a bad interval sequence, it's $O(n^2)$ in the worst case.
- With a good interval sequence, it's better than $O(n^2)$.
  - at least $O(n^{1.5})$ in the average and worst case
  - some experiments have shown average-case running times of $O(n^{1.25})$ or even $O(n^{7/6})$
- Significantly better than insertion or selection for large n:

n	n^2	$n^{1.5}$	$n^{1.25}$
10	100	31.6	17.8
100	10,000	1000	316
10,000	100,000,000	1,000,000	100,000
10^6	10^{12}	10^9	3.16×10^7

- We've wrapped insertion sort in another loop and increased its efficiency! The key is in the larger jumps that Shell sort allows.

Practicing Time Analysis

- Consider the following static method:

```
public static int mystery(int n) {
    int x = 0;
    for (int i = 0; i < n; i++) {
        x += i;          // statement 1
        for (int j = 0; j < i; j++) {
            x += j;
        }
    }
    return x;
}
```

- What is the big-O expression for the number of times that statement 1 is executed as a function of the input n?

What about now?

- Consider the following static method:

```
public static int mystery(int n) {
    int x = 0;
    for (int i = 0; i < 3*n + 4; i++) {
        x += i;          // statement 1
        for (int j = 0; j < i; j++) {
            x += j;
        }
    }
    return x;
}
```

- What is the big-O expression for the number of times that statement 1 is executed as a function of the input n?

Practicing Time Analysis

- Consider the following static method:

```
public static int mystery(int n) {
    int x = 0;
    for (int i = 0; i < n; i++) {
        x += i;          // statement 1
        for (int j = 0; j < i; j++) {
            x += j;      // statement 2
        }
    }
    return x;
}
```

- What is the big-O expression for the number of times that statement 2 is executed as a function of the input n?

value of i	number of times statement 2 is executed

Bubble Sort

- Perform a sequence of passes from left to right
  - each pass swaps adjacent elements if they are out of order
  - larger elements "bubble up" to the end of the array
- At the end of the kth pass:
  - the k rightmost elements are in their final positions
  - we don't need to consider them in subsequent passes.
- Example:

	0	1	2	3	4
	28	24	37	15	5
after the first pass:	24	28	15	5	*37*
after the second:	24	15	5	*28*	*37*
after the third:	15	5	*24*	*28*	*37*
after the fourth:	5	*15*	*24*	*28*	*37*

Implementation of Bubble Sort

```
public class Sort {
    ...
    public static void bubbleSort(int[] arr) {
        for (int i = arr.length - 1; i > 0; i--) {
            for (int j = 0; j < i; j++) {
                if (arr[j] > arr[j+1]) {
                    swap(arr, j, j+1);
                }
            }
        }
    }
}
```

- Nested loops:
  - the inner loop performs a single pass
  - the outer loop governs:
    - the number of passes (`arr.length - 1`)
    - the ending point of each pass (the current value of `i`)

Time Analysis of Bubble Sort

- Comparisons (n = length of array):
  - they are performed in the inner loop
  - *how many repetitions does each execution of the inner loop perform?*

value of i	number of comparisons
n – 1	n – 1
n – 2	n – 2
…	…
2	2
1	1

1 + 2 + … + n – 1 =

```
public static void bubbleSort(int[] arr) {
    for (int i = arr.length - 1; i > 0; i--) {
        for (int j = 0; j < i; j++) {
            if (arr[j] > arr[j+1]) {
                swap(arr, j, j+1);
            }
        }
    }
}
```

Time Analysis of Bubble Sort

- Comparisons: the kth pass performs n – k comparisons,
  so we get $C(n) = \sum_{i=1}^{n-1} i = n^2/2 - n/2 = O(n^2)$
- Moves: depends on the contents of the array
  - in the worst case:
    - $M(n) =$
  - in the best case:
- Running time:
  - $C(n)$ is always $O(n^2)$, $M(n)$ is never worse than $O(n^2)$
  - therefore, the largest term of $C(n) + M(n)$ is $O(n^2)$
- Bubble sort is a quadratic-time or $O(n^2)$ algorithm.
  - can't do much worse than bubble!

Sorting II: Divide-and-Conquer Algorithms, Distributive Sorting

Computer Science E-22
Harvard Extension School

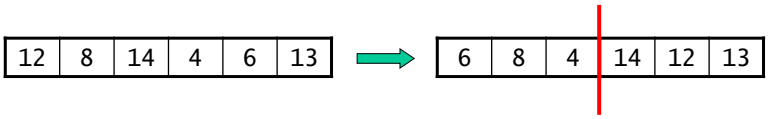
David G. Sullivan, Ph.D.

Quicksort

- Like bubble sort, quicksort uses an approach based on swapping out-of-order elements, but it's more efficient.
- A recursive, divide-and-conquer algorithm:
  - *divide:* rearrange the elements so that we end up with two subarrays that meet the following criterion:

    each element in left array <= each element in right array

    example:

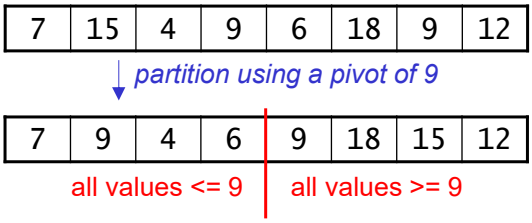


  - *conquer:* apply quicksort recursively to the subarrays, stopping when a subarray has a single element
  - *combine:* nothing needs to be done, because of the way we formed the subarrays

Partitioning an Array Using a Pivot

- The process that quicksort uses to rearrange the elements is known as *partitioning* the array.
- It uses one of the values in the array as a *pivot,* rearranging the elements to produce two subarrays:
  - left subarray: all values <= pivot
  - right subarray: all values >= pivot

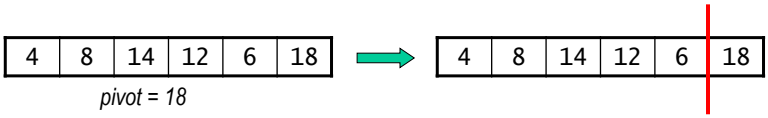
  equivalent to the criterion on the previous page.



- The subarrays will *not* always have the same length.
- This approach to partitioning is one of several variants.

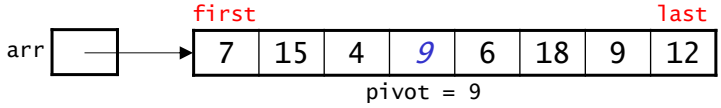
Possible Pivot Values

- First element or last element
  - risky, can lead to terrible worst-case behavior
  - especially poor if the array is almost sorted



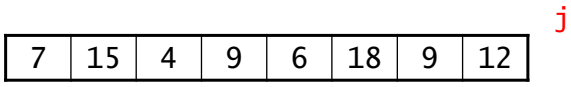
- Middle element (what we will use)
- Randomly chosen element
- Median of three elements
  - left, center, and right elements
  - three randomly selected elements
  - taking the median of three decreases the probability of getting a poor pivot

Partitioning an Array: An Example

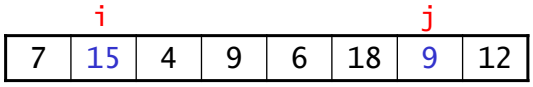


- Maintain indices `i` and `j`, starting them "outside" the array:

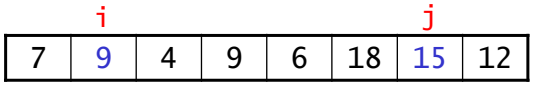
`i = first - 1`
`j = last + 1`



- *Find* "out of place" elements:
  - increment `i` until `arr[i] >= pivot`
  - decrement `j` until `arr[j] <= pivot`

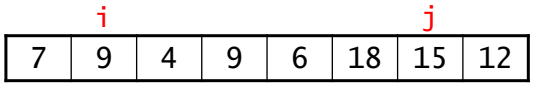


- *Swap* `arr[i]` and `arr[j]`:

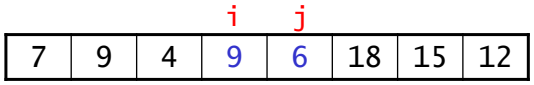


Partitioning Example (cont.)

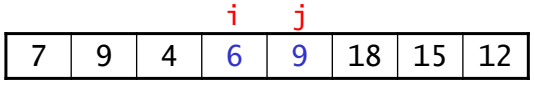
from prev. page:



- Find:



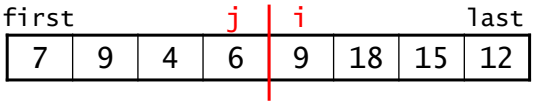
- Swap:



- Find:

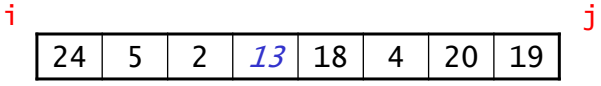
and now the indices have crossed, so we return `j`.

- Subarrays: left = from `first` to `j`, right = from `j+1` to `last`

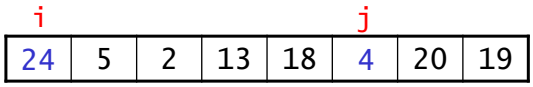


Partitioning Example 2

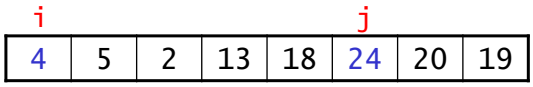
- Start (pivot = 13):



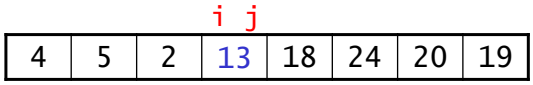
- Find:



- Swap:

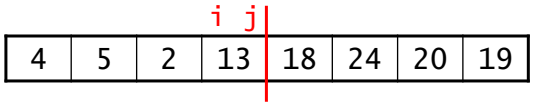


- Find:



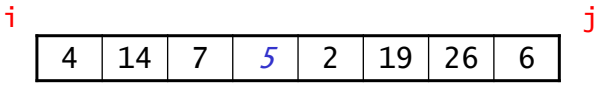
and now the indices are equal, so we return `j`.

- Subarrays:

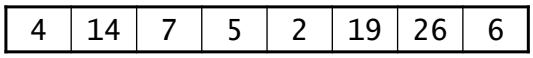


Partitioning Example 3 (done together)

- Start (pivot = 5):

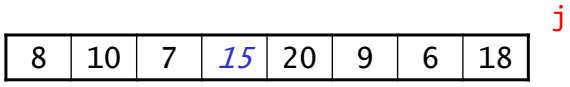


- Find:

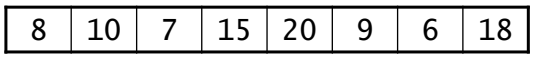


Partitioning Example 4

- Start (pivot = 15):

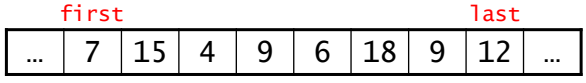


- Find:



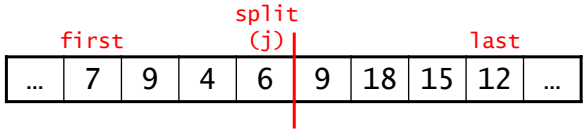
partition() Helper Method

```
private static int partition(int[] arr, int first, int last)
{
    int pivot = arr[(first + last)/2];
    int i = first - 1;  // index going left to right
    int j = last + 1;   // index going right to left
    while (true) {
        do {
            i++;
        } while (arr[i] < pivot);
        do {
            j--;
        } while (arr[j] > pivot);
        if (i < j) {
            swap(arr, i, j);
        } else {
            return j;   // arr[j] = end of left array
        }
    }
}
```



Implementation of Quicksort

```
public static void quickSort(int[] arr) { // "wrapper" method
    if (arr.length <= 1) {
        return;
    }
    qSort(arr, 0, arr.length - 1);
}

private static void qSort(int[] arr, int first, int last) {
    int split = partition(arr, first, last);

    if (first < split) {  // if left subarray has 2+ values
        qSort(arr, first, split);  // sort it recursively!
    }
    if (last > split + 1) {       // if right has 2+ values
        qSort(arr, split + 1, last);  // sort it!
    }
}   // note: base case is when neither call is made!
```



A Quick Review of Logarithms

- $\log_b n$ = the exponent to which b must be raised to get n
  - $\log_b n = p$ if $b^p = n$
  - examples: $\log_2 8 = 3$ because $2^3 = 8$
    $\log_{10} 10000 = 4$ because $10^4 = 10000$
- Another way of looking at $\log_2 n$:
  - let's say that you repeatedly divide n by 2 (using integer division)
  - $\log_2 n$ is an upper bound on the number of divisions needed to reach 1
  - example: $\log_2 18$ is approx. 4.17

    18/2 = 9 9/2 = 4 4/2 = 2 2/2 = 1

A Quick Review of Logs (cont.)

- $O(\log n)$ algorithm – one in which the number of operations is proportional to $\log_b n$ for any base b
- $\log_b n$ grows much more slowly than n

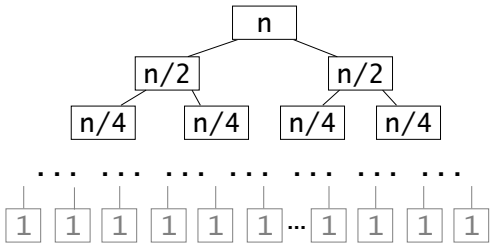
n	$\log_2 n$
2	1
1024 (1K)	10
1024*1024 (1M)	20
1024*1024*1024 (1G)	30

- Thus, for large values of n:
  - a $O(\log n)$ algorithm is much faster than a $O(n)$ algorithm
    - $\log n \ << \ n$
  - a $O(n \log n)$ algorithm is much faster than a $O(n^2)$ algorithm
    - $n * \log n \ << \ n * n$
      $n \log n \ << \ n^2$

it's also faster than a $O(n^{1.5})$ algorithm like Shell sort

Time Analysis of Quicksort

- Partitioning an array of length n requires approx. n comparisons.
  - most elements are compared with the pivot once; a few twice
- *best case:* partitioning always divides the array in half
  - repeated recursive calls give:



comparisons

n

2*(n/2) = n

4*(n/4) = n

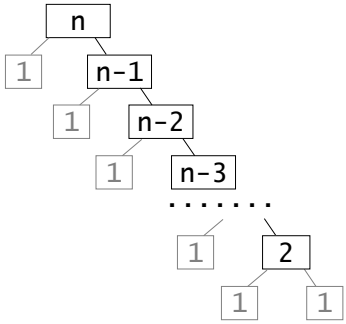
...

0

  - at each "row" except the bottom, we perform n comparisons
  - there are _____ rows that include comparisons
  - C(n) = ?
  - Similarly, M(n) and running time are both _____

Time Analysis of Quicksort (cont.)

- *worst case:* pivot is always the smallest or largest element
  - one subarray has 1 element, the other has n − 1
  - repeated recursive calls give:



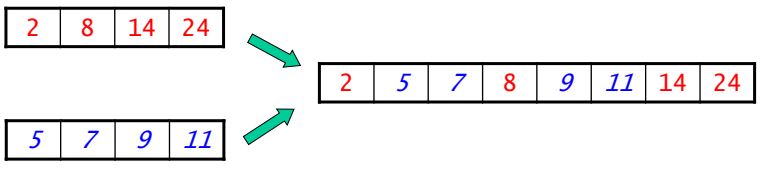
comparisons

n
n-1
n-2
n-3
...
2

  - $C(n) = \sum_{i=2}^{n} i = O(n^2)$. $M(n)$ and run time are also $O(n^2)$.
- *average case* is harder to analyze
  - $C(n) > n \log_2 n$, but it's still $O(n \log n)$

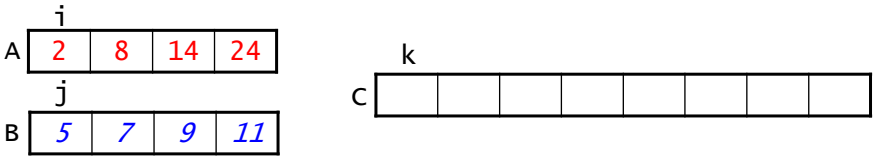
Mergesort

- The algorithms we've seen so far have sorted the array in place.
  - use only a small amount of additional memory
- Mergesort requires an additional temporary array of the same size as the original one.
  - it needs $O(n)$ additional space, where n is the array size
- It is based on the process of *merging* two sorted arrays.
  - example:

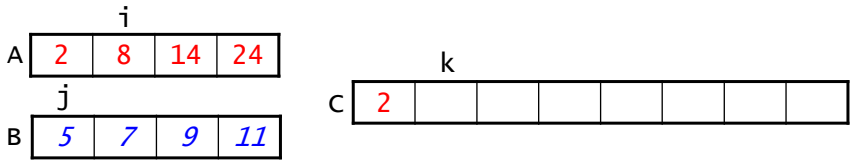


Merging Sorted Arrays

- To merge sorted arrays A and B into an array C, we maintain three indices, which start out on the first elements of the arrays:

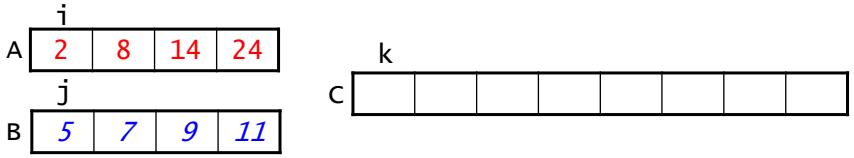


- We repeatedly do the following:
  - compare A[i] and B[j]
  - copy the smaller of the two to C[k]
  - increment the index of the array whose element was copied
  - increment k

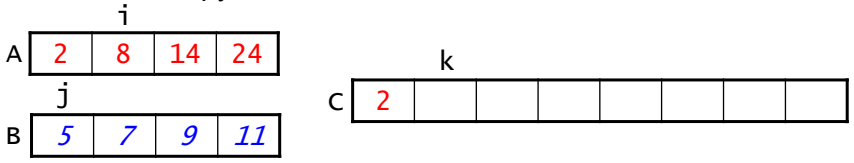


Merging Sorted Arrays (cont.)

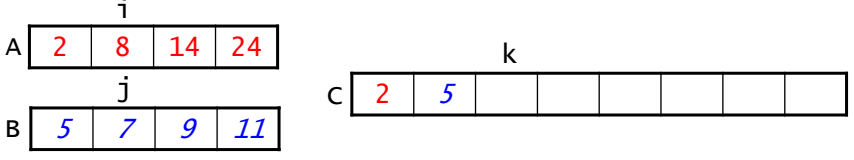
- Starting point:



- After the first copy:

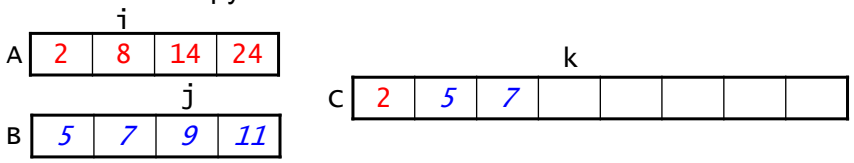


- After the second copy:

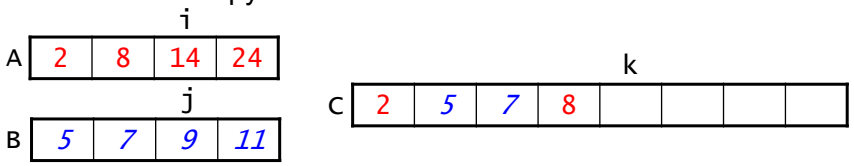


Merging Sorted Arrays (cont.)

- After the third copy:



- After the fourth copy:



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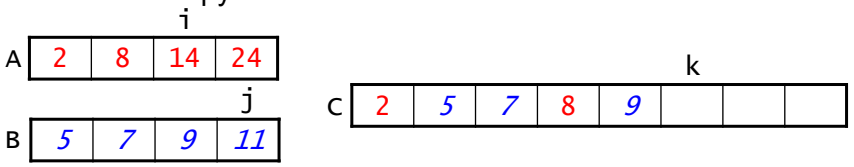
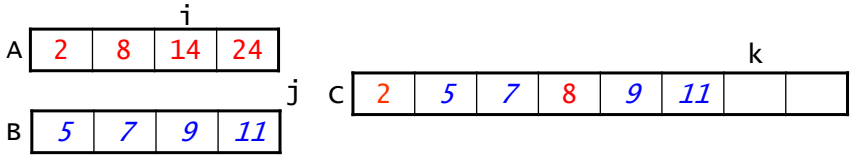
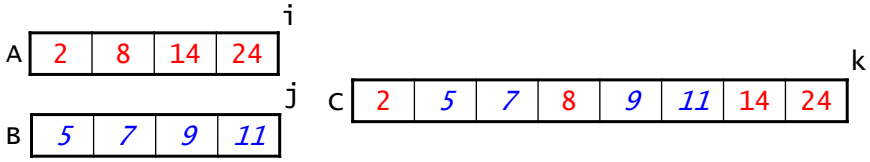


Merging Sorted Arrays (cont.)

- After the sixth copy:

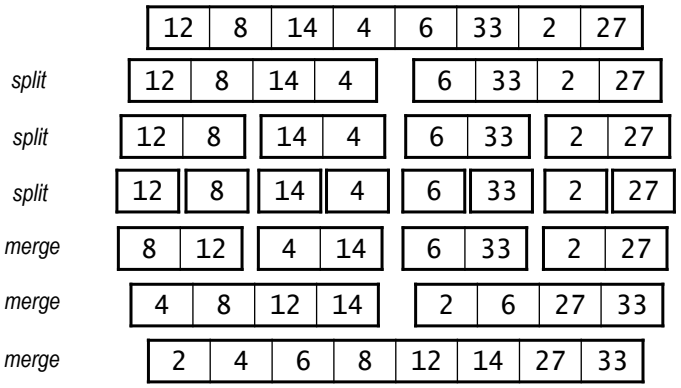


- There's nothing left in B, so we simply copy the remaining elements from A:



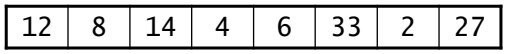
Divide and Conquer

- Like quicksort, mergesort is a divide-and-conquer algorithm.
  - *divide:* split the array in half, forming two subarrays
  - *conquer:* apply mergesort recursively to the subarrays, stopping when a subarray has a single element
  - *combine:* merge the sorted subarrays

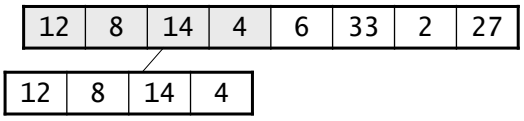


Tracing the Calls to Mergesort

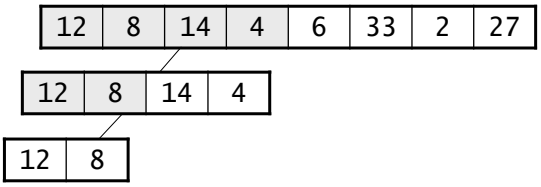
the initial call is made to sort the entire array:



split into two 4-element subarrays, and make a recursive call to sort the left subarray:

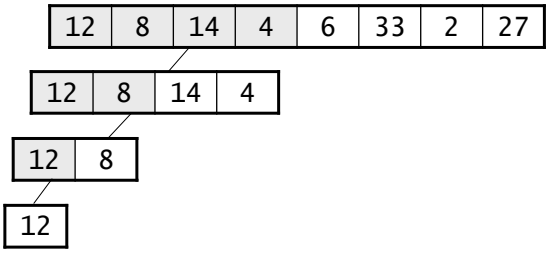


split into two 2-element subarrays, and make a recursive call to sort the left subarray:

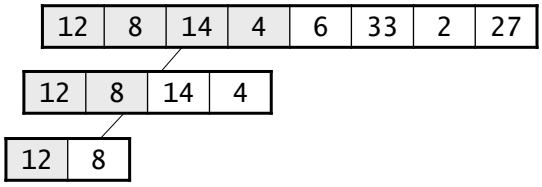


Tracing the Calls to Mergesort

split into two 1-element subarrays, and make a recursive call to sort the left subarray:

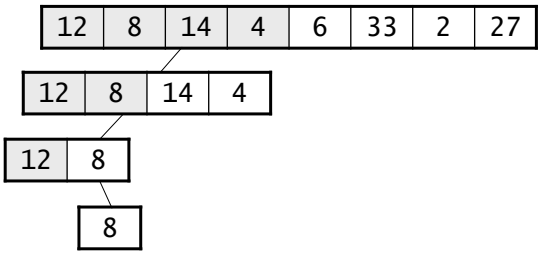


base case, so return to the call for the subarray {12, 8}:

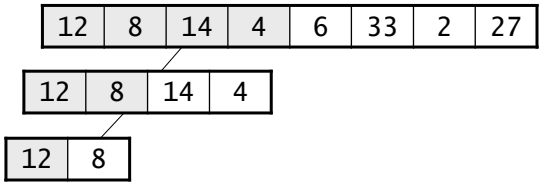


Tracing the Calls to Mergesort

make a recursive call to sort its right subarray:

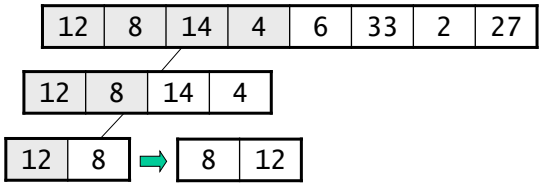


base case, so return to the call for the subarray {12, 8}:

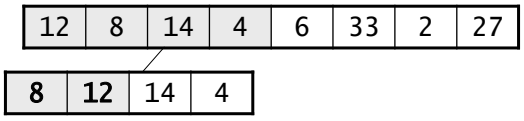


Tracing the Calls to Mergesort

merge the sorted halves of {12, 8}:

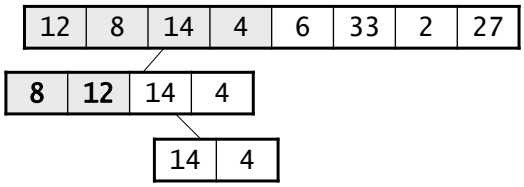


end of the method, so return to the call for the 4-element subarray, which now has a sorted left subarray:

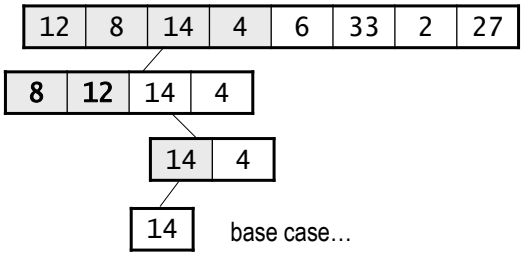


Tracing the Calls to Mergesort

make a recursive call to sort the right subarray of the 4-element subarray

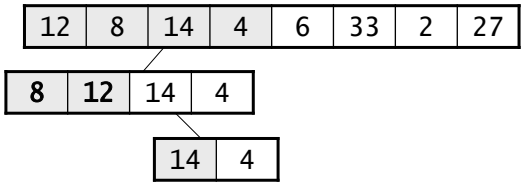


split it into two 1-element subarrays, and make a recursive call to sort the left subarray:

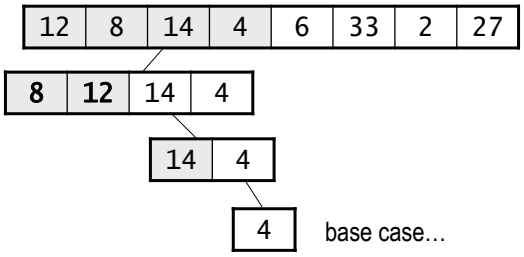


Tracing the Calls to Mergesort

return to the call for the subarray {14, 4}:

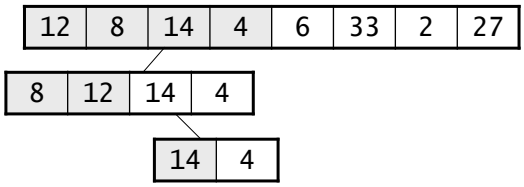


make a recursive call to sort its right subarray:

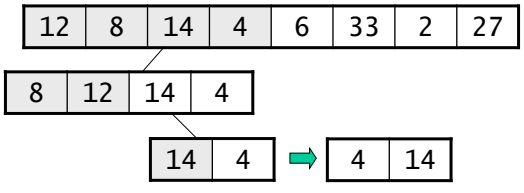


Tracing the Calls to Mergesort

return to the call for the subarray {14, 4}:

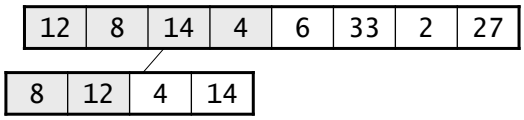


merge the sorted halves of {14, 4}:

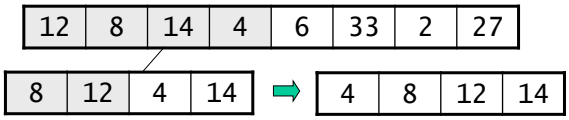


Tracing the Calls to Mergesort

end of the method, so return to the call for the 4-element subarray, which now has two sorted 2-element subarrays:



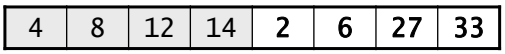
merge the 2-element subarrays:



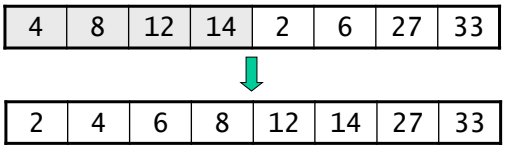
Tracing the Calls to Mergesort

end of the method, so return to the call for the original array, which now has a sorted left subarray:

perform a similar set of recursive calls to sort the right subarray. here's the result:

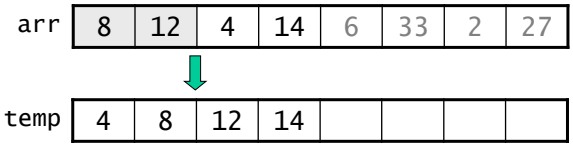


finally, merge the sorted 4-element subarrays to get a fully sorted 8-element array:

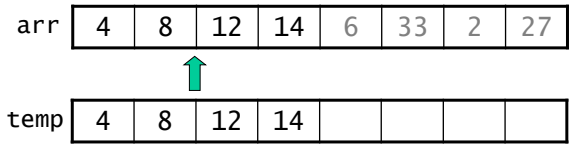


Implementing Mergesort

- In theory, we could create new arrays for each new pair of subarrays, and merge them back into the array that was split.
- Instead, we'll create a temp. array of the same size as the original.
  - pass it to each call of the recursive mergesort method
  - use it when merging subarrays of the original array:



  - after each merge, copy the result back into the original array:

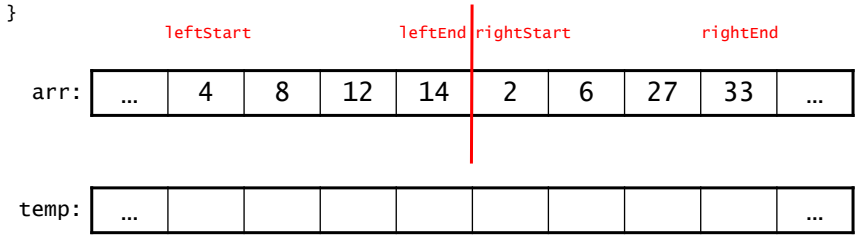


A Method for Merging Subarrays

```
private static void merge(int[] arr, int[] temp,
  int leftStart, int leftEnd, int rightStart, int rightEnd) {
     int i = leftStart;    // index into left subarray
     int j = rightStart;   // index into right subarray
     int k = leftStart;    // index into temp

     while (i <= leftEnd && j <= rightEnd) {
         if (arr[i] < arr[j]) {
             temp[k] = arr[i];
             i++; k++;
         } else {
             temp[k] = arr[j];
             j++; k++;
         }
     }

     while (i <= leftEnd) {
         temp[k] = arr[i];
         i++; k++;
     }
     while (j <= rightEnd) {
         temp[k] = arr[j];
         j++; k++;
     }

     for (i = leftStart; i <= rightEnd; i++) {
         arr[i] = temp[i];
     }
}
```

A Method for Merging Subarrays

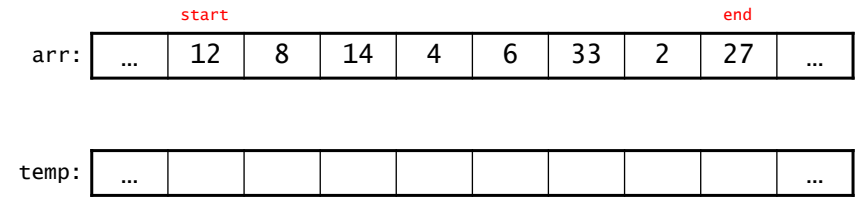
```
private static void merge(int[] arr, int[] temp,
  int leftStart, int leftEnd, int rightStart, int rightEnd) {
    int i = leftStart;    // index into left subarray
    int j = rightStart;   // index into right subarray
    int k = leftStart;    // index into temp

    while (i <= leftEnd && j <= rightEnd) { // both subarrays still have values
        if (arr[i] < arr[j]) {
            temp[k] = arr[i];
            i++; k++;
        } else {
            temp[k] = arr[j];
            j++; k++;
        }
    }

    ...
}
```



Methods for Mergesort

- Here's the key recursive method:

```
private static void mSort(int[] arr, int[] temp, int start, int end){
    if (start >= end) {  // base case: subarray of length 0 or 1
        return;
    } else {
        int middle = (start + end)/2;

        mSort(arr, temp, start, middle);
        mSort(arr, temp, middle + 1, end);

        merge(arr, temp, start, middle, middle + 1, end);
    }
}
```



Methods for Mergesort

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    } else {
        int middle = (start + end)/2;

        mSort(arr, temp, start, middle);
        mSort(arr, temp, middle + 1, end);

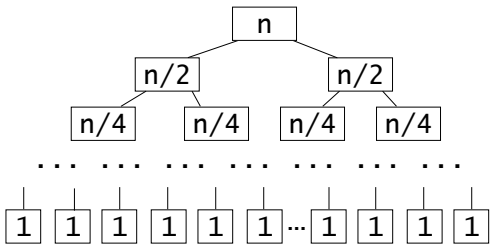
        merge(arr, temp, start, middle, middle + 1, end);
    }
}
```

- We use a "wrapper" method to create the `temp` array, and to make the initial call to the recursive method:

```
public static void mergeSort(int[] arr) {
    int[] temp = new int[arr.length];
    mSort(arr, temp, 0, arr.length - 1);
}
```

Time Analysis of Mergesort

- Merging two halves of an array of size n requires 2n moves. Why?
- Mergesort repeatedly divides the array in half, so we have the following call tree (showing the sizes of the arrays):



moves
2n
2*2*(n/2) = 2n
4*2*(n/4) = 2n
...

  - at all but the last level of the call tree, there are 2n moves
  - how many levels are there?
  - `M(n) = ?`
  - `C(n) = ?`

Summary: Sorting Algorithms

algorithm	best case	avg case	worst case	extra memory
selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$
Shell sort	$O(n \log n)$	$O(n^{1.5})$	$O(n^{1.5})$	$O(1)$
bubble sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
quicksort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	*best/avg:* $O(\log n)$ *worst:* $O(n)$
mergesort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$

- Insertion sort is best for nearly sorted arrays.
- Mergesort has the best worst-case complexity, but requires O(n) extra memory – and moves to and from the temp. array.
- Quicksort is comparable to mergesort in the best/average case.
  - efficiency is also $O(n \log n)$, but less memory and fewer moves
  - its extra memory is from…
  - with a reasonable pivot choice, its worst case is seldom seen

Comparison-Based vs. Distributive Sorting

- All of the sorting algorithms we've considered have been *comparison-based:*
  - treat the values being sorted as wholes (comparing them)
  - don't "take them apart" in any way
  - all that matters is the relative order of the values
- No comparison-based sorting algorithm can do better than $O(n \log_2 n)$ on an array of length n.
  - $O(n \log_2 n)$ is a *lower bound* for such algorithms
- *Distributive* sorting algorithms do more than compare values; they perform calculations on the values being sorted.
- Moving beyond comparisons allows us to overcome the lower bound.
  - tradeoff: use more memory.

Distributive Sorting Example: Radix Sort

- Breaks each value into a sequence of **m** components, each of which has **k** possible values.
- Examples:

	m	**k**
• integer in range 0 ... 999	3	10
• string of 15 upper-case letters	15	26
• 32-bit integer	32	2 (in binary)
	4	256 (as bytes)

- Strategy: Distribute the values into "bins" according to their last component, then concatenate the results:

33 41 12 24 31 14 13 42 34

get: 41 31 | 12 42 | 33 13 | 24 14 34

- Repeat, moving back one component each time:

get: | | |

Analysis of Radix Sort

- `m` = number of components
  `k` = number of possible values for each component
  `n` = length of the array
- Time efficiency: $O(m*n)$
  - perform m distributions, each of which processes all n values
  - $O(m*n) < O(n \log n)$ when $m < \log n$
    so we want `m` to be small
- However, there is a tradeoff:
  - as `m` decreases, `k` increases
    - fewer components ➔ more possible values per component
  - as `k` increases, so does memory usage
    - need more bins for the results of each distribution
  - increased speed requires increased memory usage

Big-*O* Notation Revisited

- We've seen that we can group functions into classes by focusing on the fastest-growing term in the expression for the number of operations that they perform.
  - e.g., an algorithm that performs $n^2/2 - n/2$ operations is a $O(n^2)$-time or quadratic-time algorithm
- Common classes of algorithms:

slower ↓

name	example expressions	big-O notation
constant time	$1, 7, 10$	$O(1)$
logarithmic time	$3\log_{10}n, \log_2 n + 5$	$O(\log n)$
linear time	$5n, 10n - 2\log_2 n$	$O(n)$
nlogn time	$4n\log_2 n, n\log_2 n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n, n^2 - 1$	$O(n^2)$
cubic time	$n^2 + 3n^3, 5n^3 - 5$	$O(n^3)$
exponential time	$2^n, 5e^n + 2n^2$	$O(c^n)$
factorial time	$3n!, 5n + n!$	$O(n!)$

How Does the Number of Operations Scale?

- Let's say that we have a problem size of 1000, and we measure the number of operations performed by a given algorithm.
- If we double the problem size to 2000, how would the number of operations performed by an algorithm increase if it is:
  - $O(n)$-time
  - $O(n^2)$-time
  - $O(n^3)$-time
  - $O(\log_2 n)$-time
  - $O(2^n)$-time

How Does the Actual Running Time Scale?

- How much time is required to solve a problem of size n?
  - assume that each operation requires 1 μsec (1×10^{-6} sec)

time function	problem size (n) 10	20	30	40	50	60
n	.00001 s	.00002 s	.00003 s	.00004 s	.00005 s	.00006 s
n^2	.0001 s	.0004 s	.0009 s	.0016 s	.0025 s	.0036 s
n^5	.1 s	3.2 s	24.3 s	1.7 min	5.2 min	13.0 min
2^n	.001 s	1.0 s	17.9 min	12.7 days	35.7 yrs	36,600 yrs

  - sample computations:
    - when n = 10, an n^2 algorithm performs 10^2 operations.
      $10^2 * (1 \times 10^{-6}$ sec) = .0001 sec
    - when n = 30, a 2^n algorithm performs 2^{30} operations.
      $2^{30} * (1 \times 10^{-6}$ sec) = 1073 sec = 17.9 min

What's the Largest Problem That Can Be Solved?

- What's the largest problem size n that can be solved in a given time T? (again assume 1 μsec per operation)

time function	time available (T) 1 min	1 hour	1 week	1 year
n	60,000,000	3.6×10^9	6.0×10^{11}	3.1×10^{13}
n^2	7745	60,000	777,688	5,615,692
n^5	35	81	227	500
2^n	25	31	39	44

  - sample computations:
    - 1 hour = 3600 sec
      that's enough time for $3600/(1 \times 10^{-6}) = 3.6 \times 10^9$ operations
      - n^2 algorithm:
        $n^2 = 3.6 \times 10^9$ → $n = (3.6 \times 10^9)^{1/2} = 60{,}000$
      - 2^n algorithm:
        $2^n = 3.6 \times 10^9$ → $n = \log_2(3.6 \times 10^9) \sim= 31$

Linked Lists

Computer Science E-22
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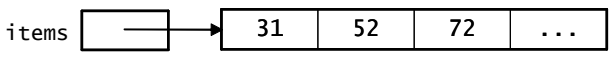
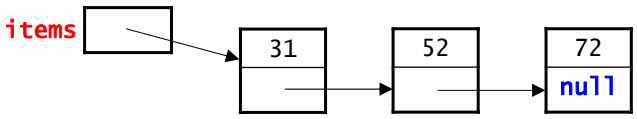
Representing a Sequence of Data

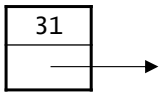
- Sequence – an ordered collection of items (position matters)
  - we will look at several types: lists, stacks, and queues
- Most common representation = an array
- Advantages of using an array:
  - easy and efficient access to *any* item in the sequence
    - `items[i]` gives you the item at position i in O(1) time
    - known as *random access*
  - very compact (but can waste space if positions are empty)
- Disadvantages of using an array:
  - have to specify an initial array size and resize it as needed
  - inserting/deleting items can require shifting other items
    - ex: insert 63 between 52 and 72



Alternative Representation: A Linked List



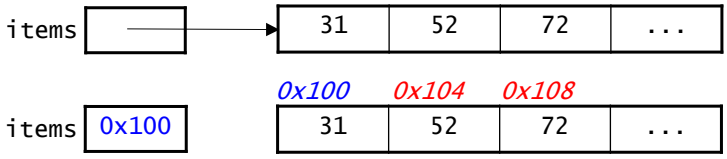
- A linked list stores a sequence of items in separate *nodes*.
- Each node is an object that contains:
  - a single item
  - a "link" (i.e., a reference) to the node containing the next item



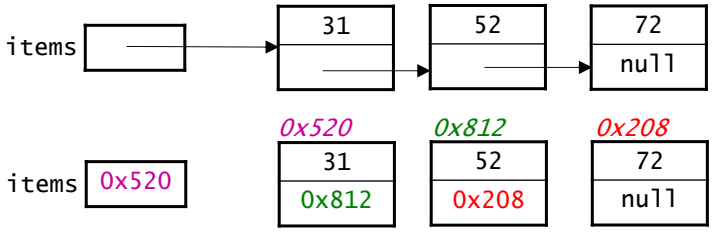
- The last node in the linked list has a link value of `null`.
- The linked list as a whole is represented by a variable that holds a reference to the first node.
  - e.g., `items` in the example above

Arrays vs. Linked Lists in Memory

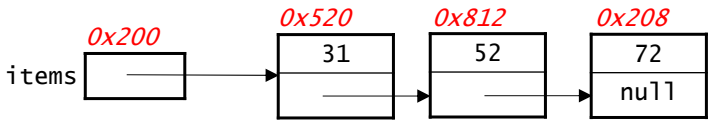
- In an array, the elements occupy consecutive memory locations:



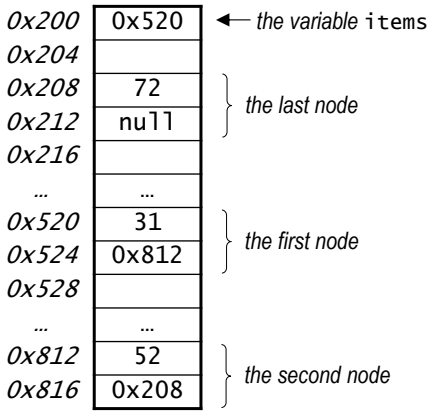
- In a linked list, the nodes are distinct objects.
  - do *not* have to be next to each other in memory
  - that's why we need the links to get from one node to the next!



Linked Lists in Memory

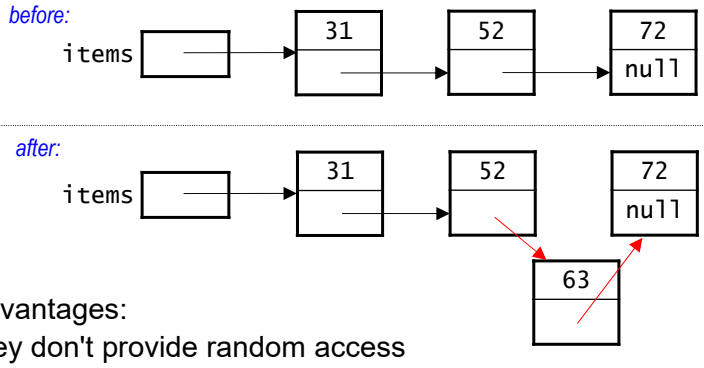


- Here's how the above linked list might actually look in memory:



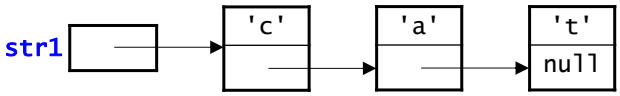
Features of Linked Lists

- They can grow without limit (provided there is enough memory).
- Easy to insert/delete an item – no need to "shift over" other items.
  - for example, to insert 63 between 52 and 72:



- Disadvantages:
  - they don't provide random access
    - need to "walk down" the list to access an item
  - the links take up additional memory

A String as a Linked List of Characters

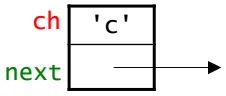


- Each node represents one character.
- Java class for this type of node:

```
public class StringNode {
    private char ch;
    private StringNode next;

    public StringNode(char c, StringNode n) {
        this.ch = c;
        this.next = n;
    }
    ...
}
```

same type as the node itself!



- The string as a whole is represented by a variable that holds a reference to the node for the first character (e.g., `str1` above).

A String as a Linked List (cont.)

- An empty string will be represented by a null value.

  example:

```
StringNode str2 = null;
```

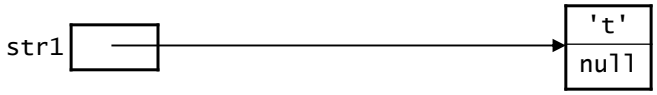
- We will use *static* methods that take the string as a parameter.
  - e.g., we'll write `length(str1)` instead of `str1.length()`
  - outside the class, call the methods using the class name:

```
StringNode.length(str1)
```

- This approach allows the methods to handle empty strings.
  - if `str1 == null`:
    - `length(str1)` will work
    - `str1.length()` will throw a `NullPointerException`

Building a Linked List of Characters I

```
public StringNode(char c,
                  StringNode n) {
    this.ch = c;
    this.next = n;
}
```

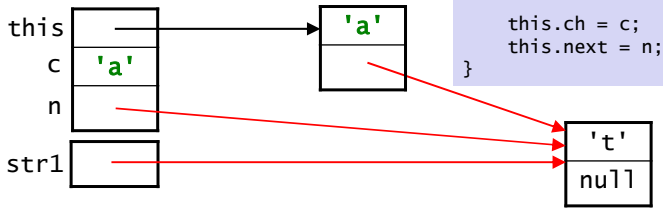


- We can use the `StringNode` constructor to build the linked list from the previous slide.
- One way is to start with the last node and work towards the front:

```
StringNode str1 = new StringNode('t', null);
```

Building a Linked List of Characters II

```
public StringNode(char c,
                  StringNode n) {
    this.ch = c;
    this.next = n;
}
```

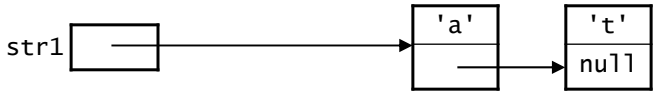


- We can use the `StringNode` constructor to build the linked list from the previous slide.
- One way is to start with the last node and work towards the front:

```
StringNode str1 = new StringNode('t', null);
str1 = new StringNode('a', str1);
```

Building a Linked List of Characters III

```
public StringNode(char c,
                  StringNode n) {
    this.ch = c;
    this.next = n;
}
```

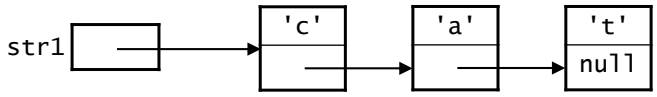


- We can use the `StringNode` constructor to build the linked list from the previous slide.
- One way is to start with the last node and work towards the front:

```
StringNode str1 = new StringNode('t', null);
str1 = new StringNode('a', str1);
```

Building a Linked List of Characters IV

```
public StringNode(char c,
                  StringNode n) {
    this.ch = c;
    this.next = n;
}
```



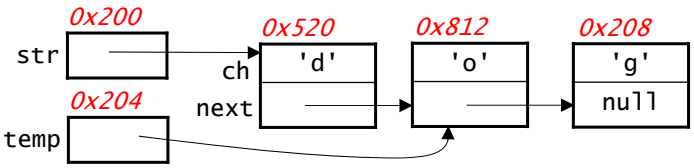
- We can use the `StringNode` constructor to build the linked list from the previous slide.
- One way is to start with the last node and work towards the front:

```
StringNode str1 = new StringNode('t', null);
str1 = new StringNode('a', str1);
str1 = new StringNode('c', str1);
```

- Later, we'll see methods that can be used to build a linked list and add nodes to it.

Review of Variables

- A variable or variable expression represents both:
  - a "box" or location in memory (the *address* of the variable)
  - the contents of that "box" (the *value* of the variable)
- Practice:

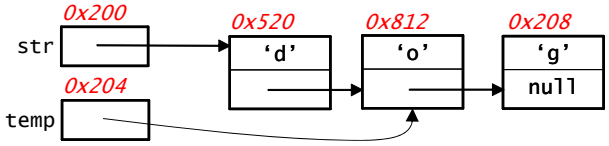



```
StringNode str;   // points to the first node
StringNode temp;  // points to the second node
```

expression	*address*	*value*
`str`	0x200	0x520 (ref to the 'd' node)
`str.ch`		
`str.next`		

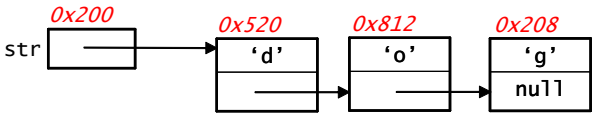
Assumptions:
- `ch` *field has the same memory address as the node itself.*
- `next` *field comes 2 bytes after the start of the node.*

More Complicated Expressions



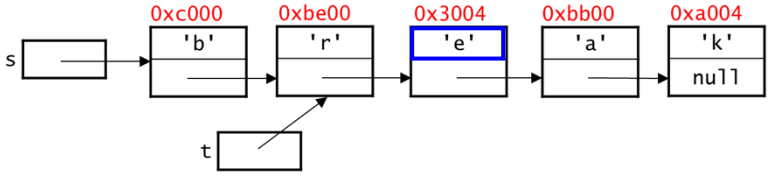
- Example: `temp.next.ch`
- Start with the beginning of the expression: `temp.next`
  It represents the `next` field of the node to which `temp` refers.
  - address =
  - value =
- Next, consider `temp.next.ch`
  It represents the `ch` field of the node to which `temp.next` refers.
  - address =
  - value =

What are the address and value of `str.next.next`?

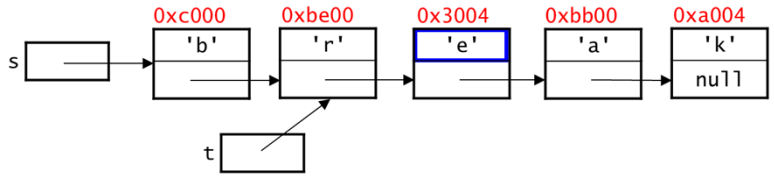


- `str.next` is…
- thus, `str.next.next` is…

What expression using `t` would give us `'e'`?



What expression using `t` would give us `'e'`?



Working backwards…

- I know that I need the `ch` field in the `'e'` node
- Where do I have a reference to the 'e' node?
- What expression can I use for the box containing that reference?

Review of Assignment Statements

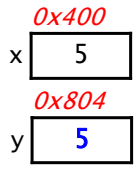
- An assignment of the form

  `var1 = var2;`

  - takes the value inside `var2`
  - copies it into `var1`

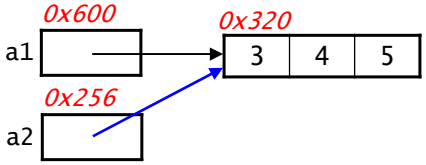
- Example involving integers:

```
int x = 5;
int y = x;
        5
```

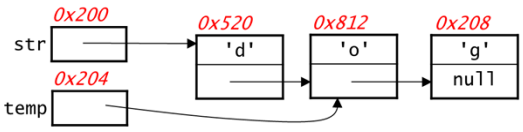


- Example involving references:

```
int[] a1 = {3, 4, 5};
int[] a2 = a1;
           0x320
```



What About These Assignments?



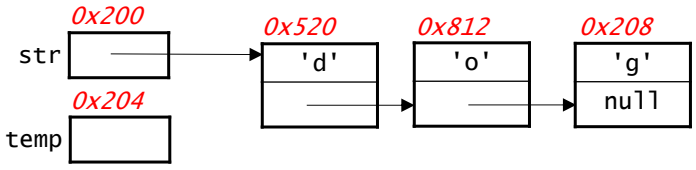
- *Identify the two boxes.*
- *Determine the value in the box specified by the right-hand side.*
- *Copy that value into the box specified by the left-hand side.*

1) `str.next = temp.next;`

2) `temp.next = temp.next.next;`

Writing an Appropriate Assignment

- If `temp` didn't already refer to the `'o'` node, what assignment would be needed to make it refer to that node?



  - start by asking: where do I currently have a reference to the `'o'` node?
  - then ask: what expression can I use for that box?
  - then write the assignment:

A Linked List Is a Recursive Data Structure!

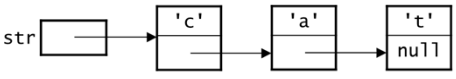
- Recursive definition: a linked list is either
  a) empty or
  b) a single node, followed by a linked list
- Viewing linked lists in this way allows us to write recursive methods that operate on linked lists.

Recursively Finding the Length of a String

- For a Java `String` object:


```
public static int length(String str) {
    if (str.equals("")) {
        return 0;
    } else {
        int lenRest = length(str.substring(1));
        return 1 + lenRest;
    }
}
```

- For a linked-list string:




```
public static int length(StringNode str) {
    if (str == null) {
        return 0;
    } else {
        int lenRest = length(str.next);
        return 1 + lenRest;
    }
}
```

An Alternative Version of the Method

- Original version:

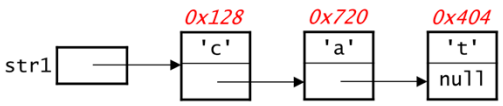
```
public static int length(StringNode str) {
    if (str == null) {
        return 0;
    } else {
        int lenRest = length(str.next);
        return 1 + lenRest;
    }
}
```

- Version without a variable for the result of the recursive call:

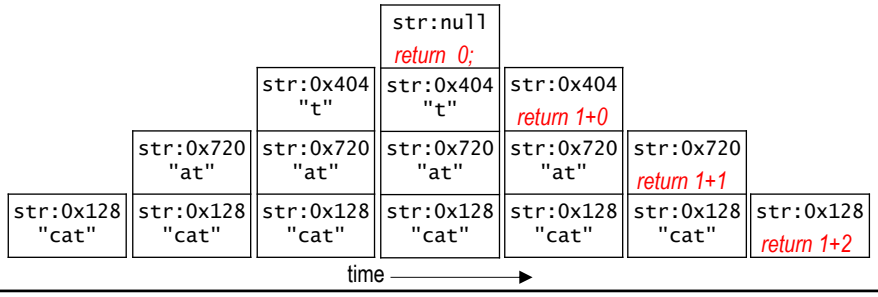
```
public static int length(StringNode str) {
    if (str == null) {
        return 0;
    } else {
        return 1 + length(str.next);
    }
}
```

Tracing `length()`

```
public static int length(StringNode str) {
    if (str == null) {
        return 0;
    } else {
        return 1 + length(str.next);
    }
}
```

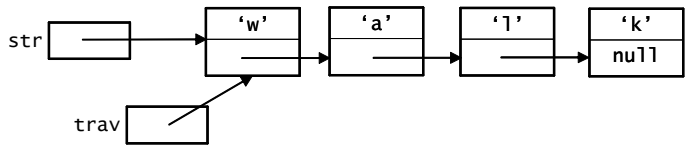


- Example: `StringNode.length(str1)`



Using Iteration to Traverse a Linked List

- Many tasks require us to traverse or "walk down" a linked list.
- We just saw a method that used recursion to do this.
- It can also be done using iteration (for loops, while loops, etc.).
- We make use of a variable (call it `trav`) that keeps track of where we are in the linked list.

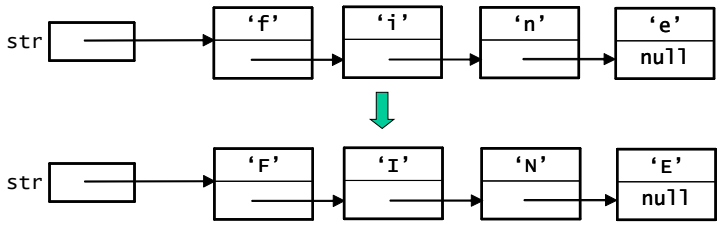


- Template for traversing an entire linked list:

```
StringNode trav = str;     // start with first node
while (trav != null) {
    // process the current node here
    trav = trav.next;      // move trav to next node
}
```

Example of Iterative Traversal

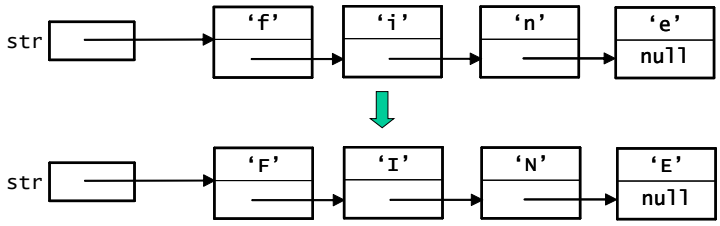
- `toUpperCase(str)`: converting `str` to all upper-case letters



- Similar to the built-in method for Java `String` objects.
- This method processes linked-list strings:
  - uses a loop to process one `StringNode` at a time
  - modifies the internals of the string (unlike the built-in version)
  - thus, it doesn't need to return anything

Example of Iterative Traversal (cont.)

- `toUpperCase(str)`: converting `str` to all upper-case letters

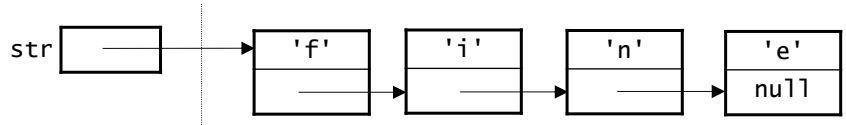


- Here's the method:

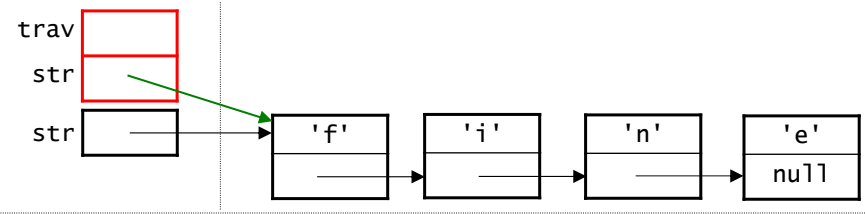
```
public static void toUpperCase(StringNode str) {
     StringNode trav = str;
     while (trav != null) {
         trav.ch = Character.toUpperCase(trav.ch);
         trav = trav.next;
     }
}
```

  - uses a built-in static method from the `Character` class to convert a single `char` to upper case

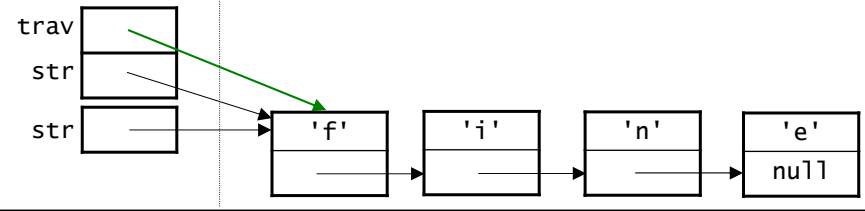
Tracing `toUpperCase()`: Before the Loop



Calling `StringNode.toUpperCase(str)` adds a stack frame to the stack:



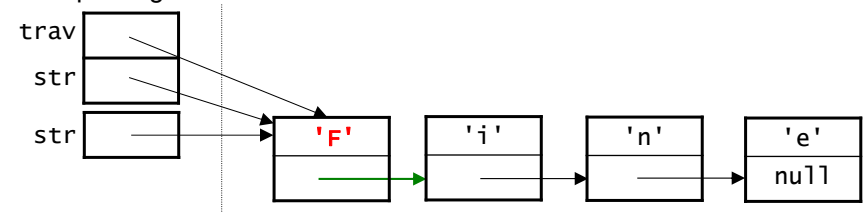
`StringNode trav = str;`



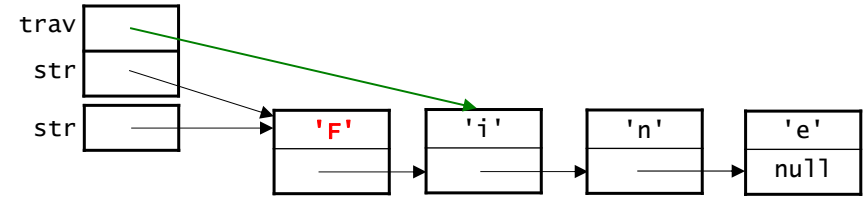
Tracing `toUpperCase()`: First Iteration of Loop

```
while (trav != null) {
    trav.ch = Character.toUpperCase(trav.ch);
    trav = trav.next;
}
```

after updating `trav.ch`:



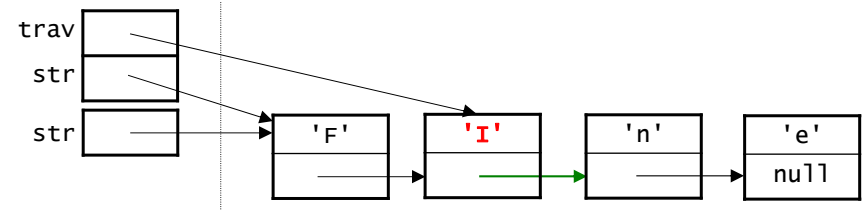
after updating `trav`:



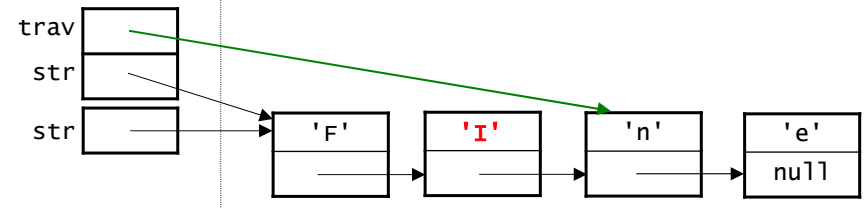
Tracing `toUpperCase()`: Second Iteration

```
while (trav != null) {
    trav.ch = Character.toUpperCase(trav.ch);
    trav = trav.next;
}
```

after updating `trav.ch`:



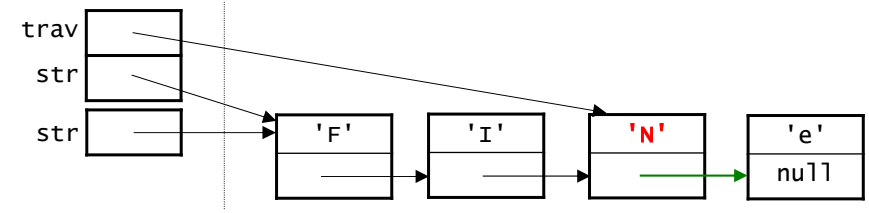
after updating `trav`:



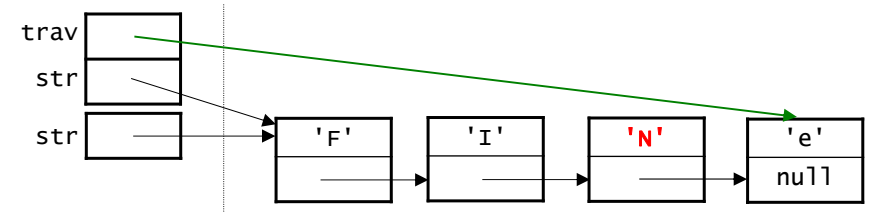
Tracing `toUpperCase()`: Third Iteration

```
while (trav != null) {
    trav.ch = Character.toUpperCase(trav.ch);
    trav = trav.next;
}
```

after updating `trav.ch`:



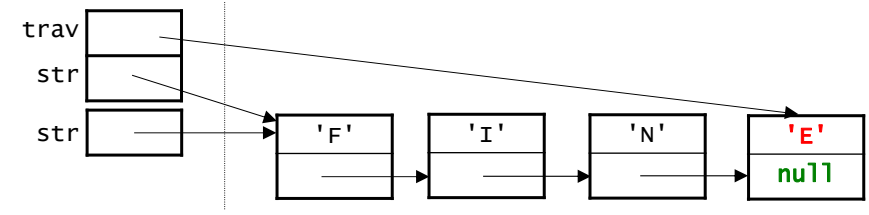
after updating `trav`:



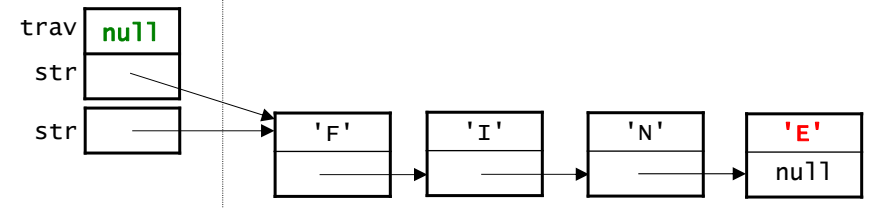
Tracing `toUpperCase()`: Fourth Iteration

```
while (trav != null) {
    trav.ch = Character.toUpperCase(trav.ch);
    trav = trav.next;
}
```

after updating `trav.ch`:



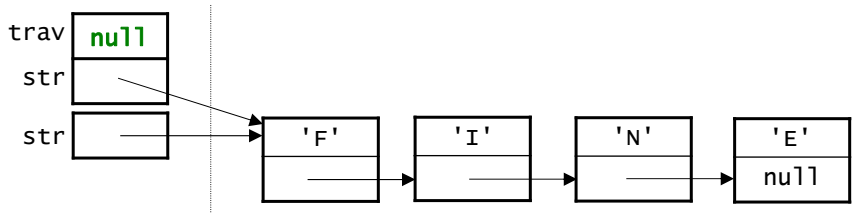
after updating `trav`:



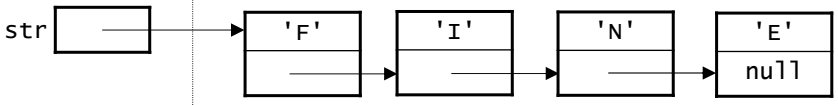
Tracing `toUpperCase()`: Finishing Up

```
while (trav != null) {
    trav.ch = Character.toUpperCase(trav.ch);
    trav = trav.next;
}
```

results of the final iteration:

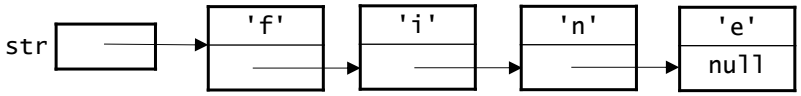


and now `trav == null`, so we end the loop and return:



Getting the Node at Position i in a Linked List

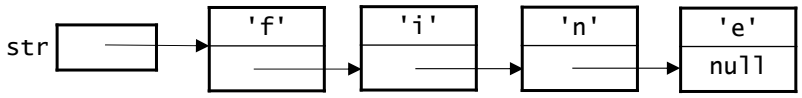
- `getNode(str, i)` – should return a reference to the `i`th node in the linked list to which `str` refers



- Examples:
  - `getNode(str, 0)` should return a ref. to the `'f'` node
  - `getNode(str, 3)` should return a ref. to the `'e'` node
  - `getNode(str.next, 2)` should return a ref. to…?

- More generally, when `0 < i <` *length of list*, `getNode(str, i)` is equivalent to `getNode(str.next, i-1)`

Getting the Node at Position i in a Linked List



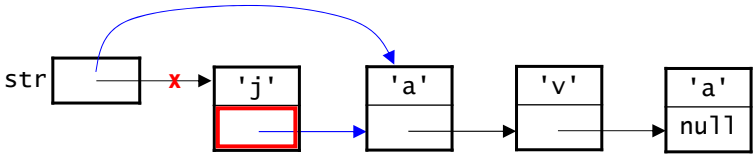
- Recursive approach to `getNode(str, i)`:
  - if `i == 0`, return `str` (base case)
  - else call `getNode(str.next, i-1)` and return what it returns!
  - other base case?
- Here's the method:

```
private static StringNode getNode(StringNode str, int i) {
    if (i < 0 || str == null) { // base case 1: no node i
        return null;
    } else if (i == 0) {        // base case 2: just found
        return str;
    } else {
        return getNode(str.next, i-1);
    }
}
```

Deleting the Item at Position i

- Special case: `i == 0` (deleting the first item)
- Update our reference to the first node by doing:

```
str = str.next;
```



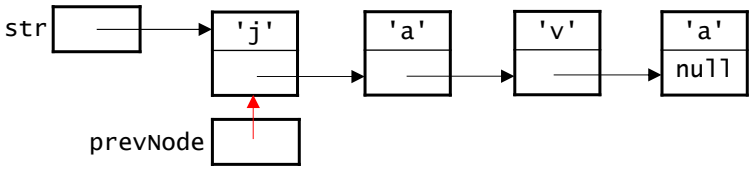
Deleting the Item at Position i (cont.)

- General case: `i > 0`

1. Obtain a reference to the *previous* node:

```
StringNode prevNode = getNode(i - 1);
```

(example for i == 1*)*



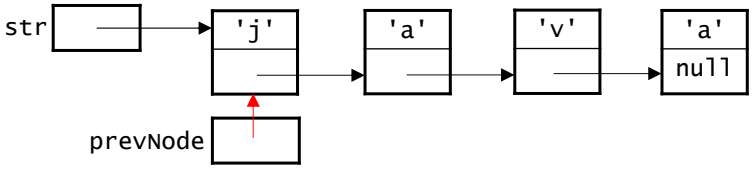
Deleting the Item at Position i (cont.)

- General case: `i > 0`

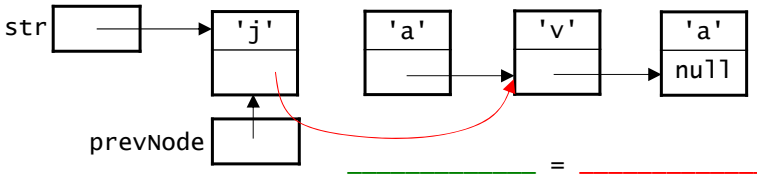
2. Update the references to remove the node

(example for i == 1*)*

before:



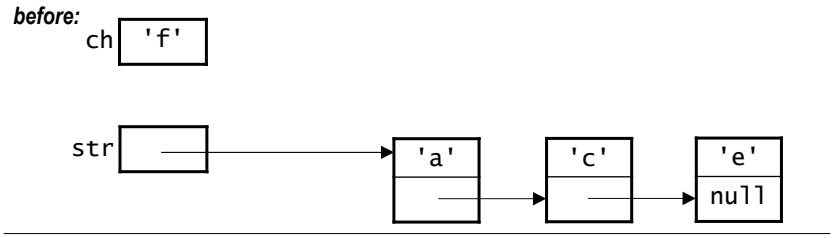
after:

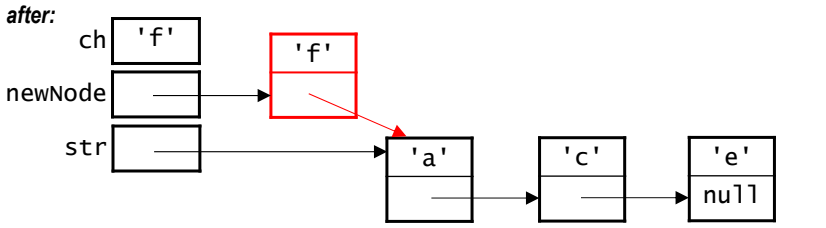


_____ = _____;

Inserting an Item at Position i

- Special case: `i == 0` (insertion at the front of the list)
- Step 1: *Create* the new node. *Fill in the blanks!*

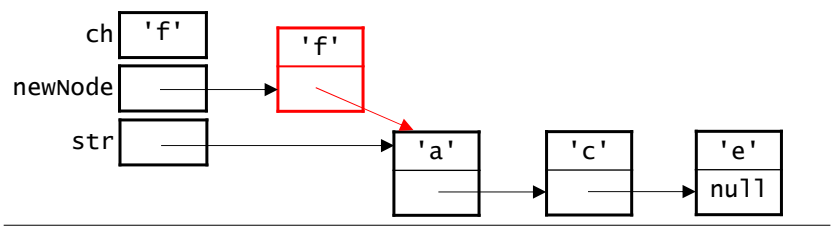


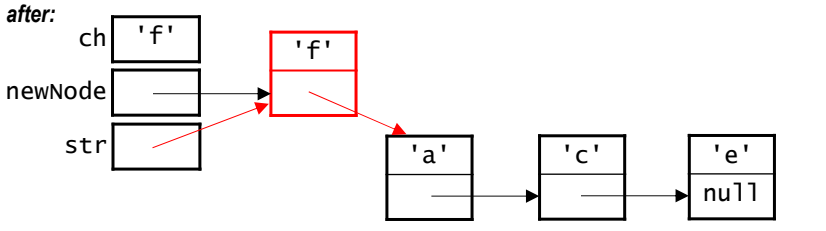



```
StringNode newNode = new StringNode(_______, _______);
```

Inserting an Item at Position i (cont.)

- Special case: `i == 0` (continued)
- Step 2: *Insert* the new node. *Write the assignment!*

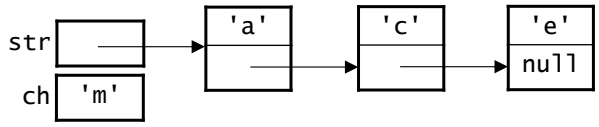




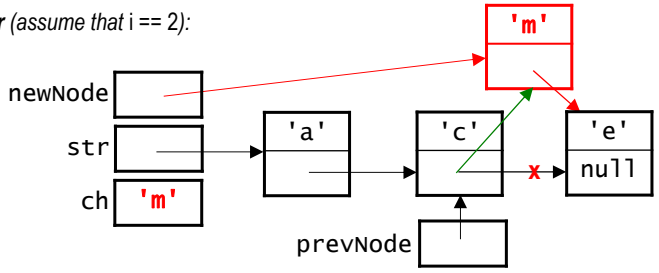
Inserting an Item at Position i (cont.)

- General case: `i > 0` (insert *before* the item currently in posn `i`)

before:



after *(assume that* i == 2*):*




```
StringNode prevNode = getNode(i - 1);
StringNode newNode = new StringNode(ch, _____________);
_____________________________ // one more line
```

Returning a Reference to the First Node

- Both `deleteChar()` and `insertChar()` return a reference to the first node in the linked list. For example:

```
public static StringNode deleteChar(StringNode str, int i) {
    …
    if (i == 0) {              // special case
        str = str.next;
    } else {                   // general case
        StringNode prevNode = getNode(str, i-1);
        if (prevNode != null && prevNode.next != null) {
            prevNode.next = prevNode.next.next;
            …
    }

    return str;
}
```

- Clients should call them as part of an assignment:

```
s1 = StringNode.deleteChar(s1, 0);
s2 = StringNode.insertChar(s2, 0, 'h');
```

- If the first node changes, the client's variable will be updated to point to the new first node.

Creating a Copy of a Linked List

- `copy(str)` – create a copy of *the entire list* to which `str` refers
- Recursive approach:
  - base case: if `str` is empty, return `null`
  - else: – make a recursive call to copy the rest of the linked list
    – create and return a copy of the first node, with its next field pointing to the copy of the rest

```
public static StringNode copy(StringNode str) {
    if (str == null) {        // base case
        return null;
    }

    // make a recursive call to copy the rest of the list
    StringNode copyRest = copy(str.next);

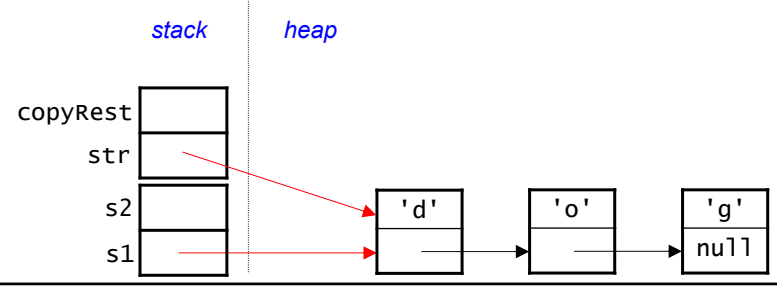
    // create and return a copy of the first node,
    // with its next field pointing to the copy of the rest
    return new StringNode(str.ch, copyRest);
}
```

Tracing copy(): the initial call

- From a client: `StringNode s2 = StringNode.copy(s1);`

```
public static StringNode copy(StringNode str) {
    if (str == null) {
        return null;
    }

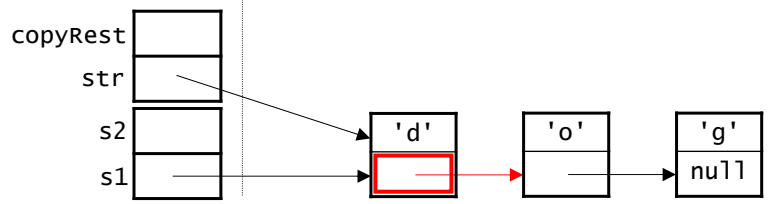
    StringNode copyRest = copy(str.next);
    return new StringNode(str.ch, copyRest);
}
```



Tracing copy(): the initial call

- From a client: `StringNode s2 = StringNode.copy(s1);`

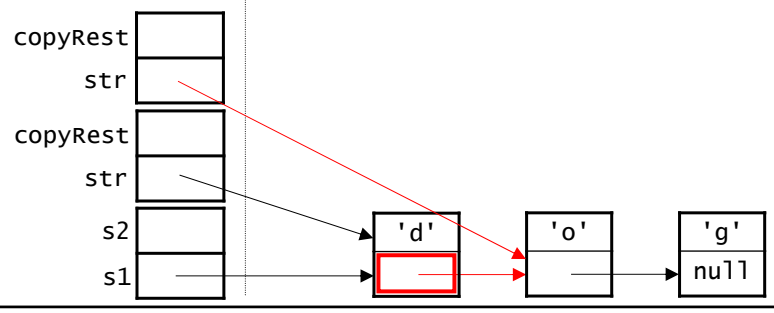
```
public static StringNode copy(StringNode str) {
    if (str == null) {
        return null;
    }
    StringNode copyRest = copy(str.next);
    return new StringNode(str.ch, copyRest);
}
```



Tracing copy(): the initial call

- From a client: `StringNode s2 = StringNode.copy(s1);`

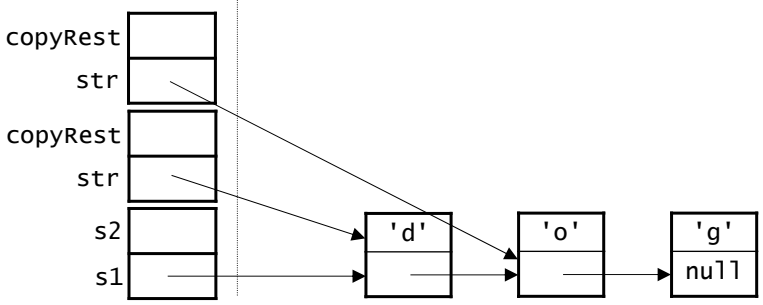
```
public static StringNode copy(StringNode str) {
    if (str == null) {
        return null;
    }
    StringNode copyRest = copy(str.next);
    return new StringNode(str.ch, copyRest);
}
```



Tracing copy(): the recursive calls

```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }

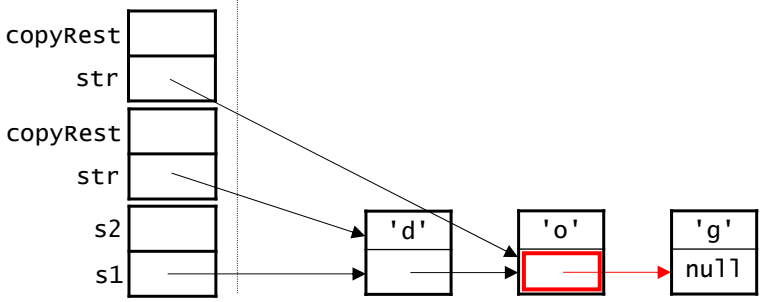
  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



Tracing copy(): the recursive calls

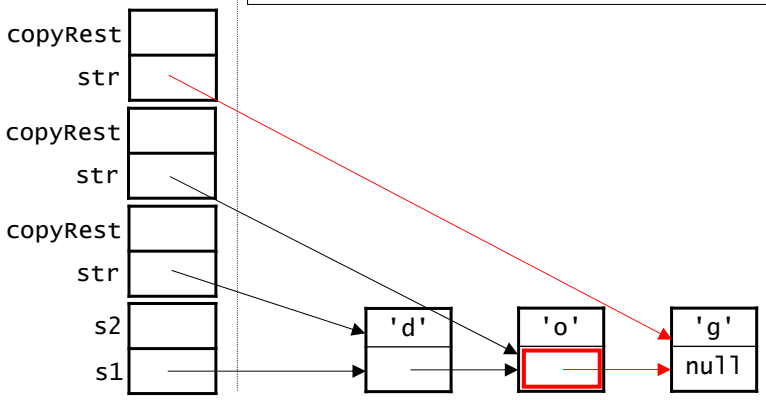
```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }

  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



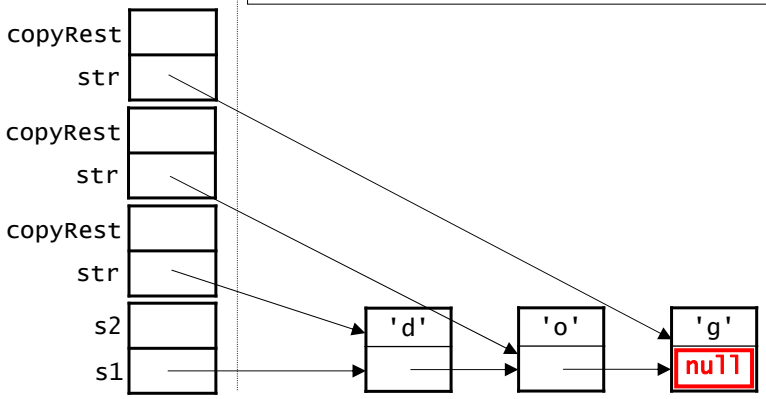
Tracing copy(): the recursive calls

```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }
  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



Tracing copy(): the recursive calls

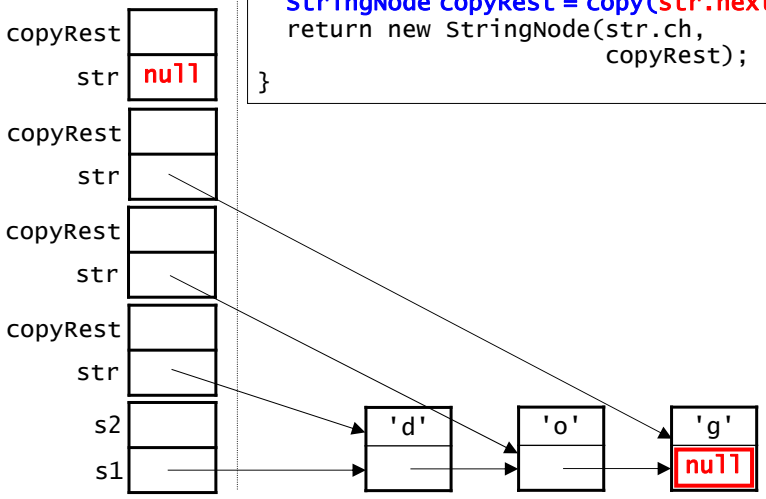
```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }
  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



Tracing copy(): the recursive calls

```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }

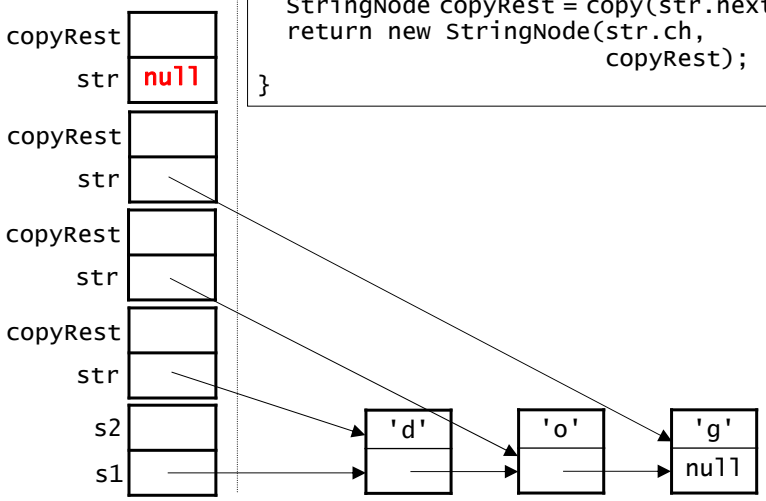
  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



Tracing copy(): the base case

```
public static StringNode copy(...) {
  if (str == null) {
      return null;
  }

  StringNode copyRest = copy(str.next);
  return new StringNode(str.ch,
                        copyRest);
}
```



Tracing copy(): returning from the base case

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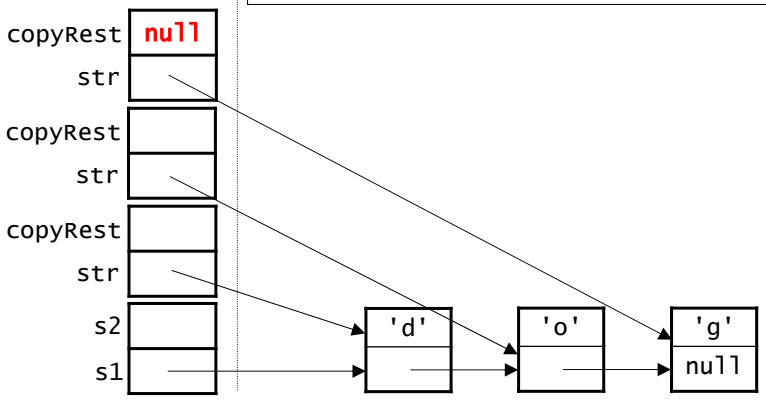

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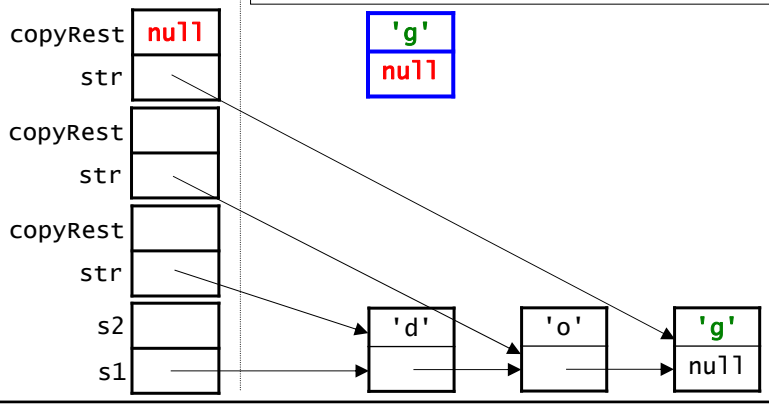

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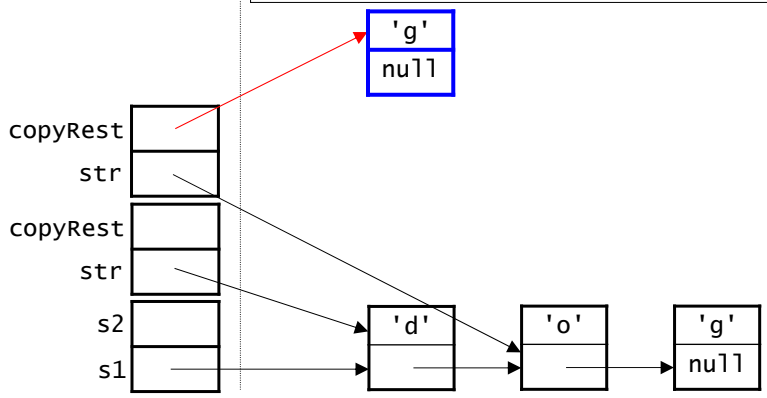

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}
```

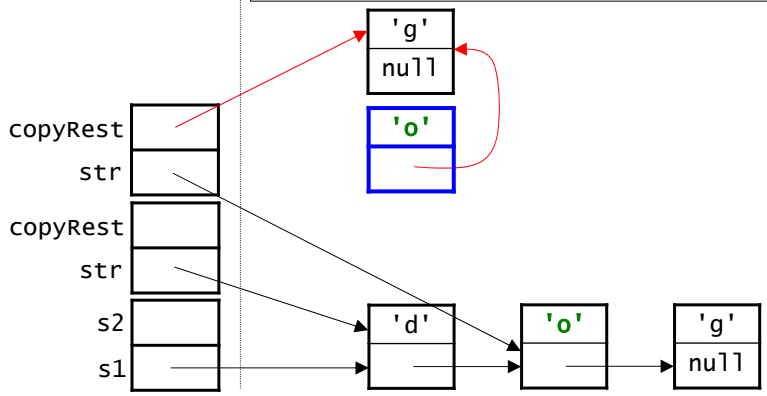

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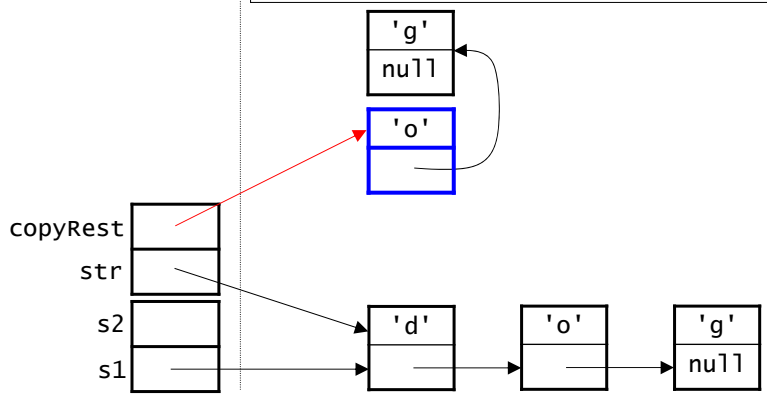

Tracing copy(): returning from the base case

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public static StringNode copy(...) {
  if (str == null) {
      return null;
  }
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                        copyRest);
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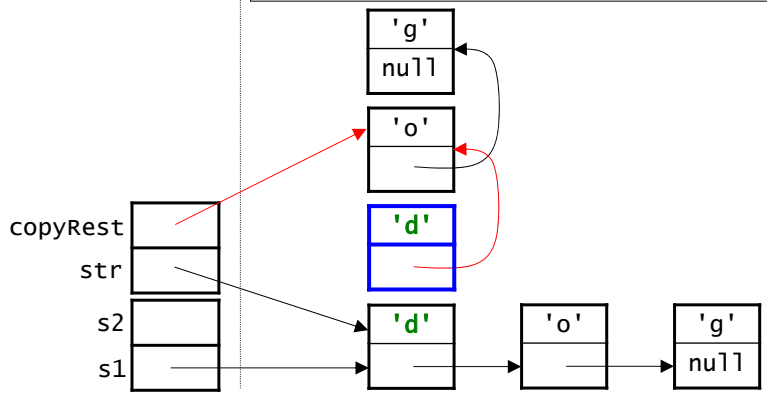
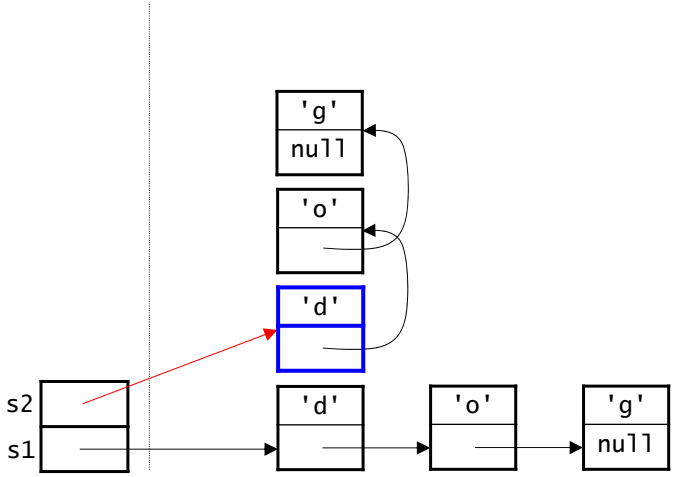
```



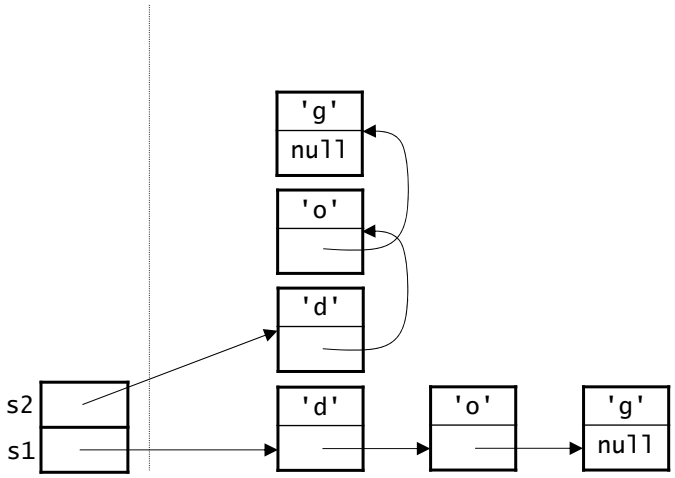
Tracing copy(): returning from the base case

- From a client: `StringNode s2 = StringNode.copy(s1);`



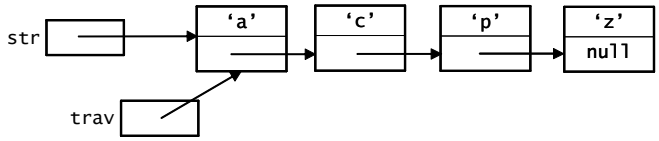
Tracing copy(): Final Result

- `s2` now holds a reference to a linked list that is a copy of the linked list to which `s1` holds a reference.



Using a "Trailing Reference" During Traversal

- When traversing a linked list, one `trav` may not be enough.
- Ex: insert `ch = 'n'` at the right place in this *sorted* linked list:



- Traverse the list to find the right position:

```
StringNode trav = str;
while (trav != null && trav.ch < ch) {
    trav = trav.next;
}
```

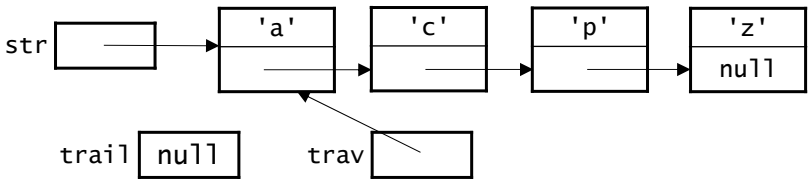
- When we exit the loop, where will `trav` point? Can we insert `'n'`?

- The following changed version doesn't work either. Why not?

```
while (trav != null && trav.next.ch < ch) {
    trav = trav.next;
}
```

Using a "Trailing Reference" (cont.)

- To get around the problem seen on the previous page, we traverse the list using two different references:
  - `trav`, which we use as before
  - `trail`, which stays one node behind `trav`




```
StringNode trav = str;
StringNode trail = null;
while (trav != null && trav.ch < ch) {
    trail = trav;
    trav = trav.next;
}
// if trail == null, insert at the front of the list
// else insert after the node to which trail refers
```

Using a "Trailing Reference" (cont.)

- To get around the problem seen on the previous page, we traverse the list using two different references:
  - `trav`, which we use as before
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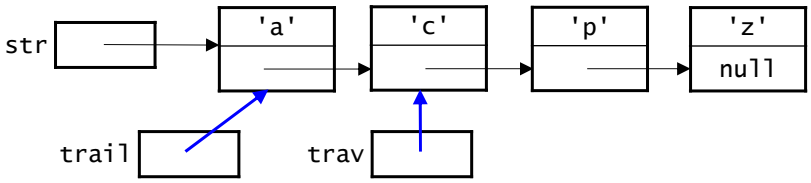


```
StringNode trav = str;
StringNode trail = null;
while (trav != null && trav.ch < ch) {
    trail = trav;
    trav = trav.next;
}
// if trail == null, insert at the front of the list
// else insert after the node to which trail refers
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Using a "Trailing Reference" (cont.)

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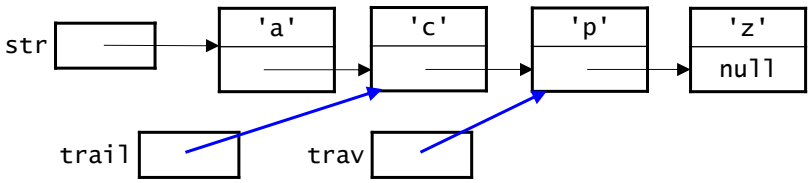
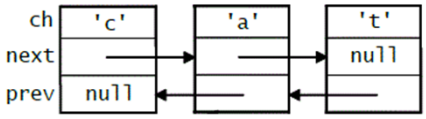


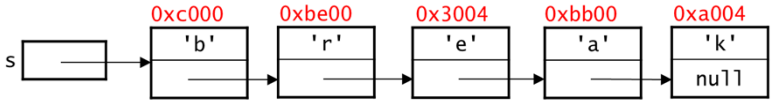

```
StringNode trav = str;
StringNode trail = null;
while (trav != null && trav.ch < ch) {
    trail = trav;
    trav = trav.next;
}
// if trail == null, insert at the front of the list
// else insert after the node to which trail refers
```

Doubly Linked Lists



- In a doubly linked list, every node stores *two* references:
  - `next`, which works the same as before
  - `prev`, which holds a reference to the previous node
    - in the first node, `prev` has a value of `null`
- The `prev` references allow us to "back up" as needed.
  - remove the need for a trailing reference during traversal!
- Insertion and deletion must update both types of references.

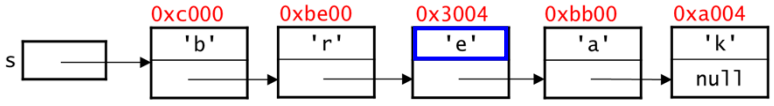
Find the address and value of `s.next.next.ch`



Extra practice!

	address	value
A.	0xbe00	`'r'`
B.	0x3004	`'e'`
C.	0xbb00	`'a'`
D.	none of these	

Find the address and value of `s.next.next.ch`



- `s.next` is the `next` field in the node to which `s` refers
  - it holds a reference to the `'r'` node
- thus, `s.next.next` is the `next` field in the `'r'` node
  - it holds a reference to the `'e'` node
- thus, `s.next.next.ch` is the `ch` field in the `'e'` node
  - it holds the `'e'`!

	address	value
A.	0xbe00	`'r'`
B.	**0x3004**	**`'e'`**
C.	0xbb00	`'a'`
D.	none of these	

Lists, Stacks, and Queues

Computer Science E-22
Harvard University

David G. Sullivan, Ph.D.

Representing a Sequence: Arrays vs. Linked Lists

- Sequence – an ordered collection of items (position matters)
  - we will look at several types: lists, stacks, and queues
- Can represent any sequence using an array *or* a linked list

	array	***linked list***
representation in memory	elements occupy consecutive memory locations	nodes can be at arbitrary locations in memory; the links connect the nodes together
advantages	• provide random access (access to any item in constant time) • no extra memory needed for links	• can grow to an arbitrary length • allocate nodes as needed • inserting or deleting does *not* require shifting items
disadvantages	• have to preallocate the memory needed for the maximum sequence size • inserting or deleting can require shifting items	• no random access (may need to traverse the list) • need extra memory for links

Abstract Data Types

- An *abstract data type* (ADT) is a model of a data structure that specifies:
  - the characteristics of the collection of data
  - the operations that can be performed on the collection
- It's *abstract* because it doesn't specify *how* the ADT will be implemented.
- A given ADT can have multiple implementations.

The List ADT

- A list is a sequence in which items can be accessed, inserted, and removed *at any position in the sequence*.
- The operations supported by our List ADT:
  - `getItem(i)`: get the item at position i
  - `addItem(item, i)`: add the specified item at position i
  - `removeItem(i)`: remove the item at position i
  - `length()`: get the number of items in the list
  - `isFull()`: test if the list already has the maximum number of items
- Note that we *don't* specify *how* the list will be implemented.

Specifying an ADT Using an Interface

- In Java, we can use an *interface* to specify an ADT:

```
public interface List {
    Object getItem(int i);
    boolean addItem(Object item, int i);
    Object removeItem(int i);
    int length();
    boolean isFull();
}
```

- An interface specifies a set of methods.
  - includes only their headers
  - does *not* typically include the full method definitions
- Like a class, it must go in a file with an appropriate name.
  - in this case: `List.java`
- Methods specified in an interface *must* be public, so we don't need the keyword `public` in the headers.

Implementing an ADT Using a Class

- To implement an ADT, we define a class.
- We specify the corresponding interface in the class header:

```
public class ArrayList implements List {
    ...
```

  - tells the compiler that the class will define *all* of the methods in the interface
  - if the class doesn't define them, it won't compile
- We'll look at two implementations of the `List` interface:
  - `ArrayList` – uses an array to store the items
  - `LLList` – uses a linked list to store the items

Recall: Polymorphism

- An object can be used wherever an object of one of its superclasses is called for.
- For example:

```
Animal a = new Dog();

Animal[] zoo = new Animal[100];
zoo[0] = new Ant();
zoo[1] = new Cat();
...
```

Another Example of Polymorphism

- An interface can be used as the type of a variable:

```
List myList;
```

- We can then assign an object of any class that implements the interface:

```
List l1 = new ArrayList(20);
List l2 = new LLList();
```

- This allows us write code that works with *any* implementation of an ADT:

```
public static void processList(List vals) {
    for (int i = 0; i < vals.length(); i++) {
      …
}
```

  - `vals` can be an object of *any* class that implements `List`
  - regardless of which class `vals` is from, we know it has all of the methods in the `List` interface

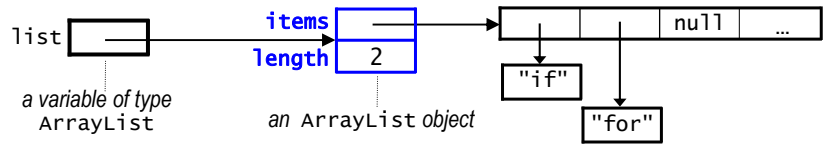
Implementing a List Using an Array

```
public class ArrayList implements List {
    private Object[] items;
    private int length;

    public ArrayList(int maxSize) {
        // code to check for invalid maxSize goes here...
        this.items = new Object[maxSize];
        this.length = 0;
    }

    public int length() {
        return this.length;
    }

    public boolean isFull() {
        return (this.length == this.items.length);
    }
    ...
}
```



Recall: The Implicit Parameter

```
public class ArrayList implements List {
    private Object[] items;
    private int length;

    public ArrayList(int maxSize) {
        this.items = new Object[maxSize];
        this.length = 0;
    }

    public int length() {
        return this.length;
    }

    public boolean isFull() {
        return (this.length == this.items.length);
    }
    ...
}
```

- All non-static methods have an implicit parameter (`this`) that refers to the called object.
- In most cases, we're allowed to omit it!
  - we'll do so in the remaining notes

Omitting The Implicit Parameter

```
public class ArrayList implements List {
    private Object[] items;
    private int length;

    public ArrayList(int maxSize) {
        items = new Object[maxSize];
        length = 0;
    }

    public int length() {
        return length;
    }

    public boolean isFull() {
        return (length == items.length);
    }
    ...
}
```

- In a non-static method, if we use a variable that
  - isn't declared in the method
  - has the name of one of the fields

  Java assumes that we're using the field.

Adding an Item to an `ArrayList`

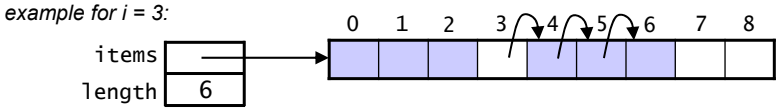
- Adding at position i (shifting items i, i+1, … to the right by one):

```
public boolean addItem(Object item, int i) {
    if (item == null || i < 0 || i > length) {
        throw new IllegalArgumentException();
    } else if (isFull()) {
        return false;
    }

    // make room for the new item
    for (int j = length - 1; j >= i; j--) {
        items[j + 1] = items[j];
    }

    items[i] = item;
    length++;
    return true;
}
```

example for i = 3:



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    for (int j = length - 1; j >= i; j--) {
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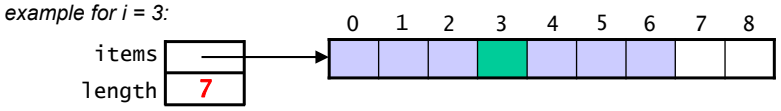
    }

    items[i] = item;
    length++;
    return true;
}
```

example for i = 3:



Removing an Item from an `ArrayList`

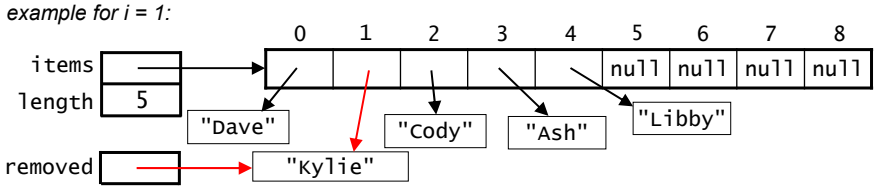
- Removing item i (shifting items i+1, i+2, … to the left by one):

```
public Object removeItem(int i) {
    if (i < 0 || i >= length) {
        throw new IndexOutOfBoundsException();
    }
    Object removed = items[i];

    // shift items after items[i] to the left
    for (int j = i; j < length - 1; j++) {
        ____________________________;
    }
    items[length - 1] = null;

    length--;
    return removed;
}
```

example for i = 1:



Getting an Item from an `ArrayList`

- Getting item i (without removing it):

```
public Object getItem(int i) {
    if (i < 0 || i >= length) {
        throw new IndexOutOfBoundsException();
    }
    return items[i];
}
```

`toString()` Method for the `ArrayList` Class

```
public String toString() {
    String str = "{";

    if (length > 0) {
        for (int i = 0; i < length - 1; i++) {
            str = str + items[i] + ", ";
        }
        str = str + items[length - 1];
    }

    str = str + "}";

    return str;
}
```

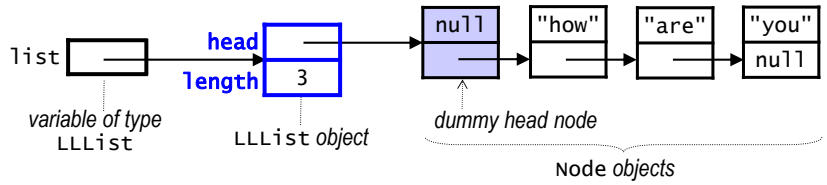
- Produces a string of the following form:

```
{items[0], items[1], … }
```

- Why is the last item added outside the loop?
- Why do we need the `if` statement?

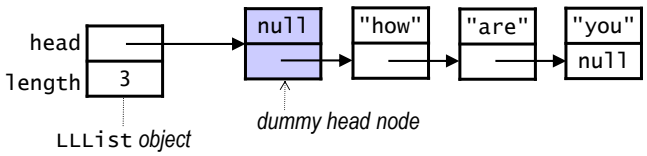
Implementing a List Using a Linked List

```
public class LLList implements List {
    private Node head;
    private int length;
    ...
}
```

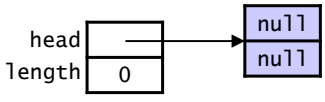


- Differences from the linked lists we used for strings:
  - we "embed" the linked list inside another class
    - users of our `LLList` class won't actually touch the nodes
  - we use non-static methods instead of static ones
    `myList.length()` instead of `length(myList)`
  - we use a special *dummy head node* as the first node

Using a Dummy Head Node



- The dummy head node is always at the front of the linked list.
  - like the other nodes in the linked list, it's of type `Node`
  - it does *not* store an item
  - it does *not* count towards the length of the list
- Using it allows us to avoid special cases when adding and removing nodes from the linked list.
- An empty `LLList` still has a dummy head node:

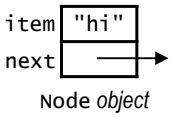


An Inner Class for the Nodes

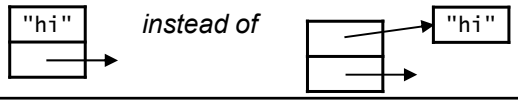
```
public class LLList implements List {
    private class Node {
        private Object item;
        private Node next;

        private Node(Object i, Node n) {
            item = i;
            next = n;
        }
    }
    ...
}
```

private since only LLList will use it



- We make `Node` an *inner class*, defining it within `LLList`.
  - allows the `LLList` methods to directly access `Node`'s private fields, while restricting access from outside `LLList`
  - the compiler creates this class file: `LLList$Node.class`
- For simplicity, our diagrams may show the items inside the nodes.



Other Details of Our `LLList` Class

```
public class LLList implements List {
    private class Node {
        // see previous slide
    }

    private Node head;
    private int length;

    public LLList() {
        head = new Node(null, null);
        length = 0;
    }

    public boolean isFull() {
        return false;
    }
    ...
}
```

- Unlike `ArrayList`, there's no need to preallocate space for the items. The constructor simply creates the dummy head node.
- The linked list can grow indefinitely, so the list is never full!

Getting a Node

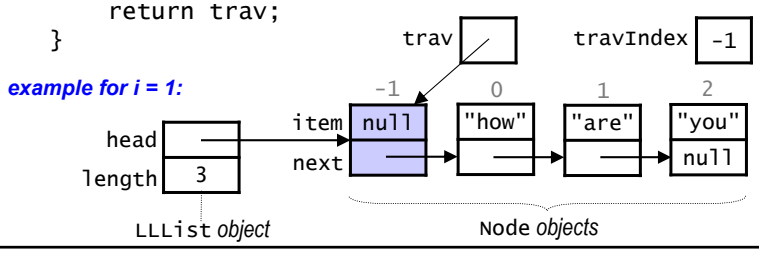
- Private helper method for getting node `i`
  - to get the dummy head node, use `i = -1`

```
private Node getNode(int i) {
    // private method, so we assume i is valid!

    Node trav = ____________;
    int travIndex = -1;
    while ( ____________________ ) {
        travIndex++;

        ____________________________;
    }

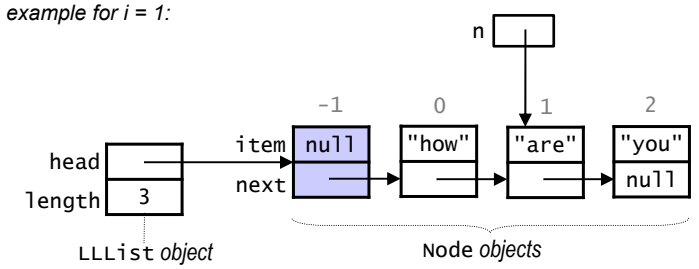
    return trav;
}
```



Getting an Item

```
public Object getItem(int i) {
    if (i < 0 || i >= length) {
        throw new IndexOutOfBoundsException();
    }

    Node n = getNode(i);
    return _________;
}
```

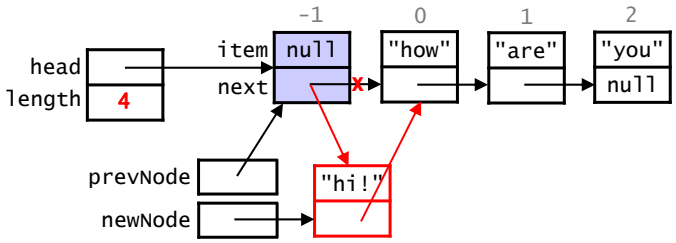


Adding an Item to an `LLList`

```
public boolean addItem(Object item, int i) {
    if (item == null || i < 0 || i > length) {
        throw new IllegalArgumentException();
    }
    Node newNode = new Node(item, null);
    Node prevNode = getNode(i - 1);
    newNode.next = prevNode.next;
    prevNode.next = newNode;

    length++;
    return true;
}
```

- This works even when adding at the front of the list (`i = 0`):



`addItem()` Without a Dummy Head Node

```
public boolean addItem(Object item, int i) {
    if (item == null || i < 0 || i > length) {
        throw new IllegalArgumentException();
    }
    Node newNode = new Node(item, null);

    if (i == 0) {                  // case 1: add to front
        newNode.next = head;
        head = newNode;
    } else {                       // case 2: i > 0
        Node prevNode = getNode(i - 1);
        newNode.next = prevNode.next;
        prevNode.next = newNode;
    }

    length++;
    return true;
}
```

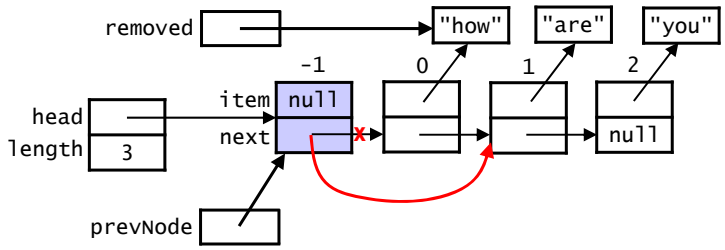
(the gray code shows what we would need to add if we didn't have a dummy head node)

Removing an Item from an `LLList`

```
public Object removeItem(int i) {
    if (i < 0 || i >= length) {
        throw new IndexOutOfBoundsException();
    }
    Node prevNode = getNode(i - 1);
    Object removed = prevNode.next.item;
                                    // what line goes here?

    length--;
    return removed;
}
```

- This works even when removing the first item (`i = 0`):



`toString()` Method for the `LLList` Class

```
public String toString() {
    String str = "{";

    // what should go here?









    str = str + "}";

    return str;
}
```

Efficiency of the List ADT Implementations

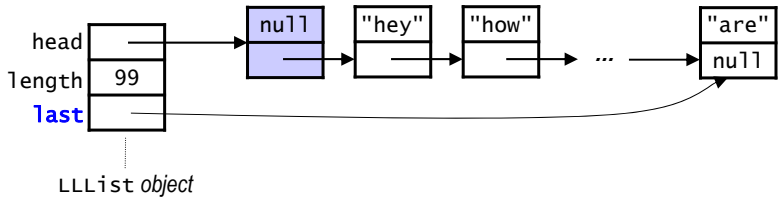
n = number of items in the list

	`ArrayList`	`LLList`
`getItem()`	only one case:	***best:*** ***worst:*** ***average:***
`addItem()`	***best:*** ***worst:*** ***average:***	***best:*** ***worst:*** ***average:***

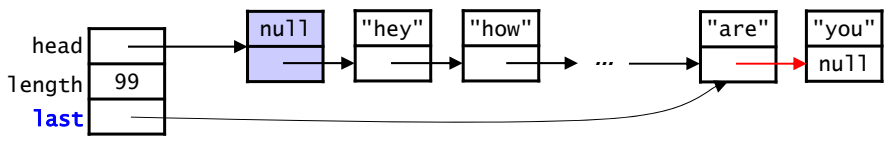
Example of Using a Reference to the Last Node

```
mylist.addItem("you", 99)
```

- before the call is made:



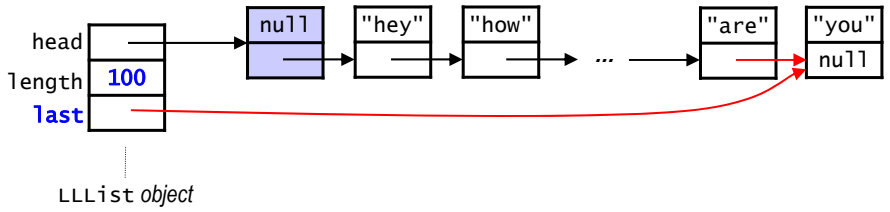
- use `last` to add the new item's node to the end of the linked list:



Example of Using a Reference to the Last Node (cont.)

```
mylist.addItem("you", 99)
```

- after the call is made:



Efficiency of the List ADT Implementations (cont.)

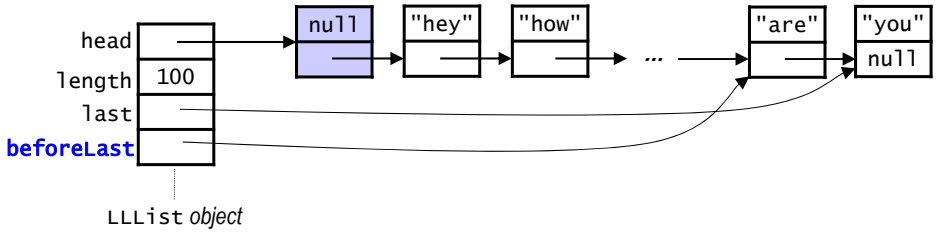
n = number of items in the list

	`ArrayList`	`LLList`
`removeItem()`	***best:*** ***worst:*** ***average:***	***best:*** ***worst:*** ***average:***
space efficiency		

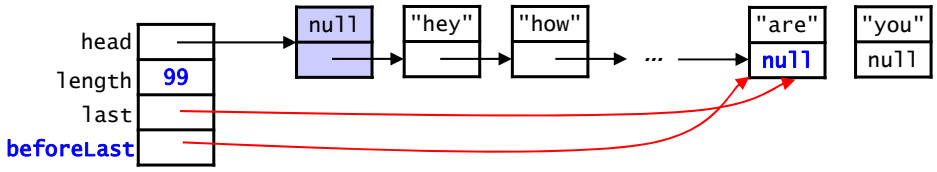
A Reference to the Second-to-Last Node Doesn't Help

```
mylist.removeItem(99)
```

- before the call is made:



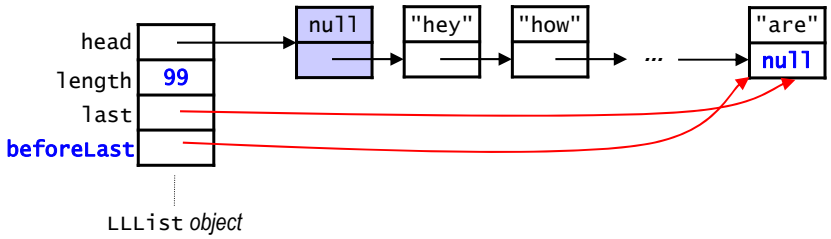
- we can use `beforeLast` to remove the last node and update `last`:



A Reference to the Second-to-Last Node Doesn't Help

```
mylist.removeItem(99)
```

- but in order to update `beforeLast`, we need to walk down the linked list!



Counting the Number of Occurrences of an Item

```
public class MyClass {
    public static int numOccur(List l, Object item) {
        int numOccur = 0;
        for (int i = 0; i < l.length(); i++) {
            Object itemAt = l.getItem(i);
            if (itemAt.equals(item)) {
                numOccur++;
            }
        }
        return numOccur;
    } ...
```

- This method works fine if we pass in an `ArrayList` object.
  - time efficiency (as a function of the length, n) = ?
- However, it's *not* efficient if we pass in an `LLList`.
  - each call to `getItem()` calls `getNode()`
  - to access item 0, `getNode()` accesses 2 nodes (dummy + node 0)
  - to access item 1, `getNode()` accesses 3 nodes
  - to access item i, `getNode()` accesses i+2 nodes
  - 2 + 3 + … + (n+1) = ?

Solution: Provide an Iterator

```
public class MyClass {
    public static int numOccur(List l, Object item) {
        int numOccur = 0;
        ListIterator iter = l.iterator();
        while (iter.hasNext()) {
            Object itemAt = iter.next();
            if (itemAt.equals(item)) {
                numOccur++;
            }
        }
        return numOccur;
    } ...
```

- We add an `iterator()` method to the `List` interface.
  - it returns a separate *iterator object* that can efficiently iterate over the items in the list
- The iterator has two key methods:
  - `hasNext()`: tells us if there are items we haven't seen yet
  - `next()`: returns the next item *and* advances the iterator

An Interface for List Iterators

- Here again, the interface only includes the method headers:

```
public interface ListIterator { // in ListIterator.java
    boolean hasNext();
    Object next();
}
```

- We can then implement this interface for each type of list:
  - `LLListIterator` for an iterator that works with `LLLists`
  - `ArrayListIterator` for an iterator for `ArrayLists`
- We use the interfaces when declaring variables in client code:

```
public class MyClass {
    public static int numOccur(List l, Object item) {
        int numOccur = 0;
        ListIterator iter = l.iterator();
        ...
```

  - doing so allows the code to work for any type of list!

Using an Inner Class for the Iterator

```
public class LLList {
    private Node head;
    private int length;

    private class LLListIterator implements ListIterator {
        private Node nextNode;  // points to node with the next item

        public LLListIterator() {
            nextNode = head.next;  // skip over dummy head node
        }
        ...
    }

    public ListIterator iterator() {
        return new LLListIterator();
    }
    ...
```

- Using an inner class gives the iterator access to the list's internals.
- The `iterator()` method is an `LLList` method.
  - it creates an instance of the inner class and returns it
  - its return type is the interface type
    - so it will work in the context of client code

Full `LLListIterator` Implementation

```
private class LLListIterator implements ListIterator {
    private Node nextNode;     // points to node with the next item

    public LLListIterator() {
        nextNode = head.next;  // skip over the dummy head node
    }

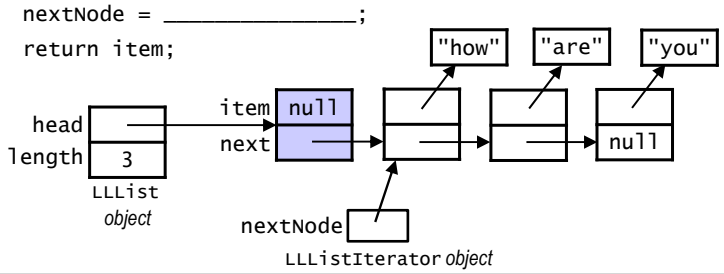
    public boolean hasNext() {
        return (nextNode != null);
    }

    public Object next() {
        // throw an exception if nextNode is null

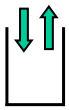
        Object item = _____________;

        nextNode = _____________;

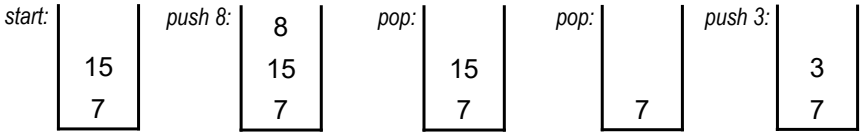
        return item;
    }
}
```



Stack ADT



- A stack is a sequence in which:
  - items can be added and removed only at one end (the *top*)
  - you can only access the item that is currently at the top
- Operations:
  - push: add an item to the top of the stack
  - pop: remove the item at the top of the stack
  - peek: get the item at the top of the stack, but don't remove it
  - isEmpty: test if the stack is empty
  - isFull: test if the stack is full
- Example: a stack of integers



A Stack Interface: First Version

```
public interface Stack {
    boolean push(Object item);
    Object pop();
    Object peek();
    boolean isEmpty();
    boolean isFull();
}
```

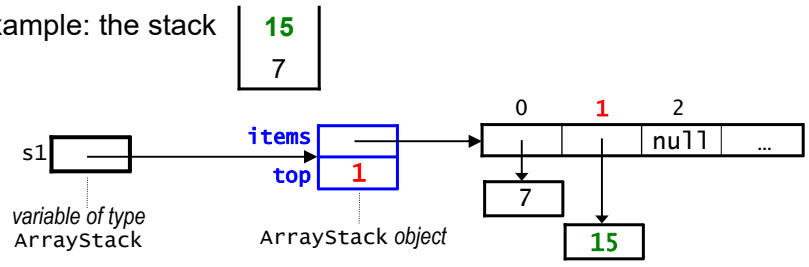
- `push()` returns `false` if the stack is full, and `true` otherwise.
- `pop()` and `peek()` take no arguments, because we know that we always access the item at the top of the stack.
  - return `null` if the stack is empty.
- The interface provides no way to access/insert/delete an item at an arbitrary position.
  - encapsulation allows us to ensure that our stacks are only manipulated in appropriate ways

Implementing a Stack Using an Array: First Version

```
public class ArrayStack implements Stack {
    private Object[] items;
    private int top;    // index of the top item

    public ArrayStack(int maxSize) {
        // code to check for invalid maxSize goes here...
        items = new Object[maxSize];
        top = -1;
    }
    ...
```

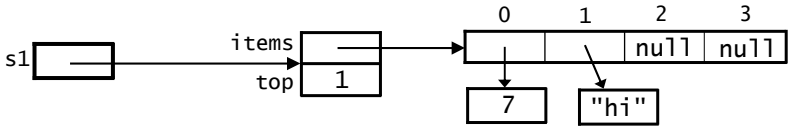
- Example: the stack



- Items are added from left to right (top item = the rightmost one).
  - `push()` and `pop()` won't require any shifting!

Collection Classes and Data Types

```
public class ArrayStack implements Stack {
    private Object[] items;
    private int top;    // index of the top item
    ...
}
```



- So far, our collections have allowed us to add objects of any type.

```
ArrayStack s1 = new ArrayStack(4);
s1.push(7);    // 7 is turned into an Integer object for 7
s1.push("hi");
String item = s1.pop();           // won't compile
String item = (String)s1.pop();   // need a type cast
```

- We'd like to be able to limit a given collection to one type.

```
ArrayStack<String> s2 = new ArrayStack<String>(10);
s2.push(7);               // won't compile
s2.push("hello");
String item = s2.pop();   // no cast needed!
```

Limiting a Stack to Objects of a Given Type

- We can do this by using a *generic* interface and class.
- Here's a generic version of our `Stack` interface:

```
public interface Stack<T> {
    boolean push(T item);
    T pop();
    T peek();
    boolean isEmpty();
    boolean isFull();
}
```

- It includes a *type variable* `T` in its header and body.
  - used as a placeholder for the actual type of the items

A Generic ArrayStack Class

```
public class ArrayStack<T> implements Stack<T> {
    private T[] items;
    private int top;    // index of the top item
    …
    public boolean push(T item) {
        …
    }
    …
}
```

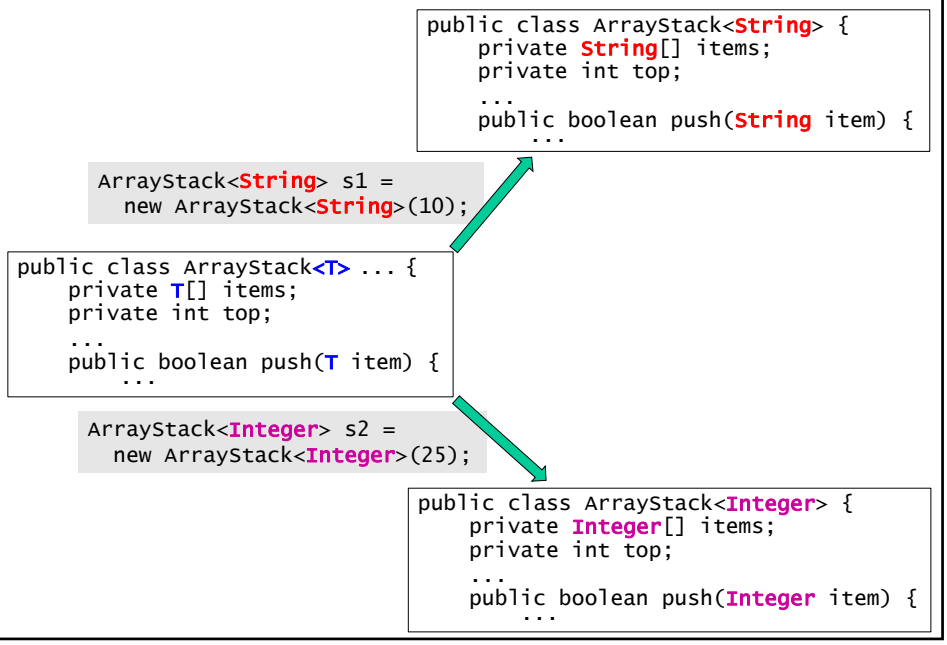
- Once again, a type variable `T` is used as a placeholder for the actual type of the items.
- When we create an `ArrayStack`, we specify the type of items that we intend to store in the stack:

```
ArrayStack<String> s1 = new ArrayStack<String>(10);
ArrayStack<Integer> s2 = new ArrayStack<Integer>(25);
```

- We can still allow for a mixed-type collection:

```
ArrayStack<Object> s3 = new ArrayStack<Object>(20);
```

Using a Generic Class



ArrayStack Constructor

- Java doesn't allow you to create an object or array using a type variable. Thus, we *cannot* do this:

```
public ArrayStack(int maxSize) {
    // code to check for invalid maxSize goes here...
    items = new T[maxSize];   // not allowed
    top = -1;
}
```

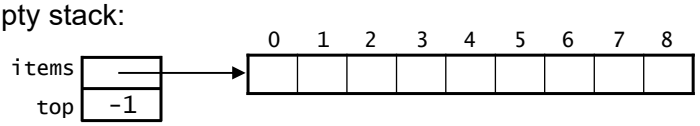
- Instead, we do this:

```
public ArrayStack(int maxSize) {
    // code to check for invalid maxSize goes here...
    items = (T[])new Object[maxSize];
    top = -1;
}
```

- The cast generates a compile-time warning, but we'll ignore it.
- Java's built-in `ArrayList` class takes this same approach.

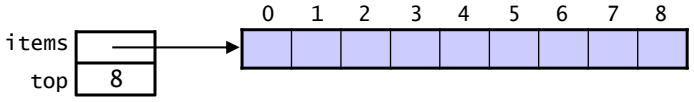
Testing if an ArrayStack is Empty or Full

- Empty stack:



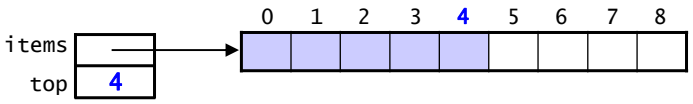

```
public boolean isEmpty() {
    return (top == -1);
}
```

- Full stack:



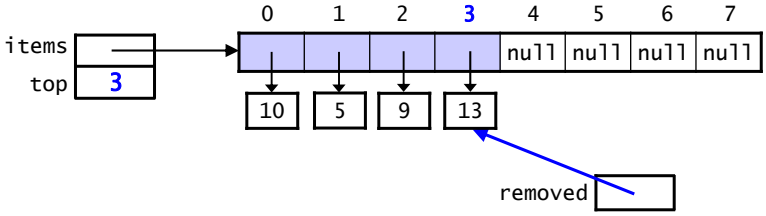

```
public boolean isFull() {
    return (top == items.length - 1);
}
```

Pushing an Item onto an ArrayStack




```
public boolean push(T item) {
    // code to check for a null item goes here
    if (isFull()) {
        return false;
    }
    top++;
    items[top] = item;
    return true;
}
```

ArrayStack pop() and peek()




```
public T pop() {
    if (isEmpty()) {
        return null;
    }

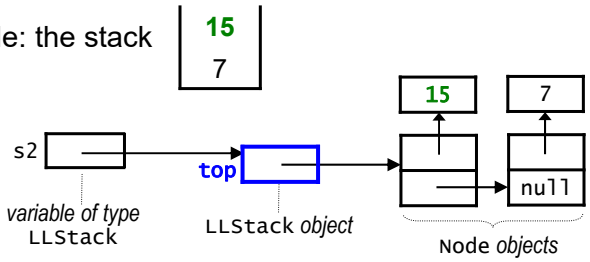
    ______ removed = items[top];
    items[top] = null;
    top--;
    return removed;
}
```

- peek just returns `items[top]` without decrementing `top`.

Implementing a Generic Stack Using a Linked List

```
public class LLStack<T> implements Stack<T> {
    private Node top;    // top of the stack
    …
}
```

- Example: the stack



- Things worth noting:
  - our `LLStack` class needs only a single field:
    a reference to the first node, which holds the top item
  - top item = leftmost item (vs. rightmost item in `ArrayStack`)
  - we don't need a dummy node
    - only one case: always insert/delete at the front of the list!

Other Details of Our `LLStack` Class

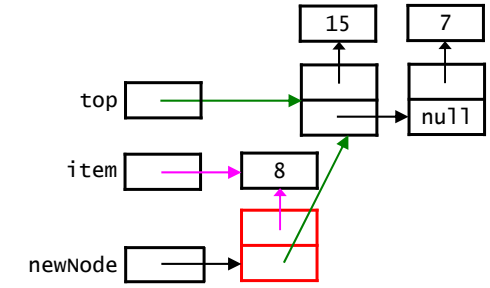
```
public class LLStack<T> implements Stack<T> {
    private class Node {
        private T item;
        private Node next;
        ...
    }

    private Node top;

    public LLStack() {
        top = null;
    }
    public boolean isEmpty() {
        return (top == null);
    }
    public boolean isFull() {
        return false;
    }
}
```

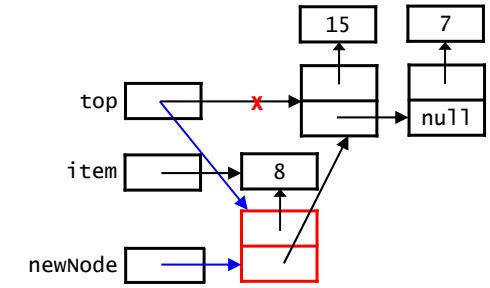
- The inner `Node` class uses the type parameter `T` for the item.
- We don't need to preallocate any memory for the items.
- The stack is never full!

LLStack push()



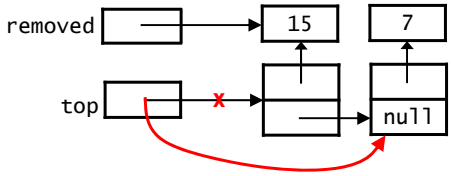

```
public boolean push(T item) {
    // code to check for a null item goes here
    Node newNode = new Node(item, top);
    top = newNode;
    return true;
}
```

LLStack push()




```
public boolean push(T item) {
    // code to check for a null item goes here
    Node newNode = new Node(item, top);
    top = newNode;
    return true;
}
```

LLStack pop() and peek()




```
public T pop() {
    if (isEmpty()) {
        return null;
    }
    T removed = ________________;

    ____________________________;
    return removed;
}
public T peek() {
    if (isEmpty()) {
        return null;
    }
    return top.item;
}
```

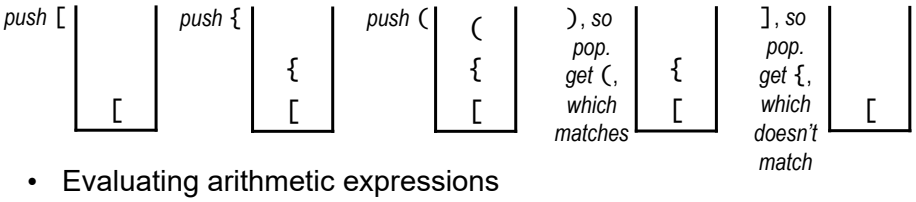
Efficiency of the Stack Implementations

	ArrayStack	**LLStack**
`push()`	*O*(1)	*O*(1)
`pop()`	*O*(1)	*O*(1)
`peek()`	*O*(1)	*O*(1)
space efficiency	*O*(m) where m is the *anticipated* maximum number of items	*O*(n) where n is the number of items currently on the stack

Applications of Stacks

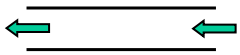
- Converting a recursive algorithm to an iterative one
  - use a stack to emulate the runtime stack
- Making sure that delimiters (parens, brackets, etc.) are balanced:
  - push open (i.e., left) delimiters onto a stack
  - when you encounter a close (i.e., right) delimiter, pop an item off the stack and see if it matches
  - example:

```
5 * [3 + {(5 + 16 - 2)]
```



- Evaluating arithmetic expressions

Queue ADT



- A queue is a sequence in which:
  - items are added at the rear and removed from the front
    - first in, first out (FIFO) (vs. a stack, which is last in, first out)
  - you can only access the item that is currently at the front
- Operations:
  - insert: add an item at the rear of the queue
  - remove: remove the item at the front of the queue
  - peek: get the item at the front of the queue, but don't remove it
  - isEmpty: test if the queue is empty
  - isFull: test if the queue is full
- Example: a queue of integers

start: `12 8`
insert 5: `12 8 5`
remove: `8 5`

Our Generic Queue Interface

```
public interface Queue<T> {
    boolean insert(T item);
    T remove();
    T peek();
    boolean isEmpty();
    boolean isFull();
}
```

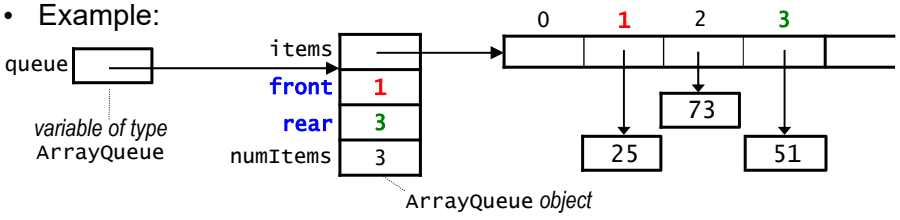
- `insert()` returns `false` if the queue is full, and `true` otherwise.
- `remove()` and `peek()` take no arguments, because we always access the item at the front of the queue.
  - return `null` if the queue is empty.
- Here again, we will use encapsulation to ensure that the data structure is manipulated only in valid ways.

Implementing a Queue Using an Array

```
public class ArrayQueue<T> implements Queue<T> {
    private T[] items;
    private int front;
    private int rear;
    private int numItems;

    ...
}
```

- Example:

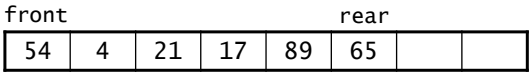


- We maintain two indices:
  - `front`: the index of the item at the front of the queue
  - `rear`: the index of the item at the rear of the queue

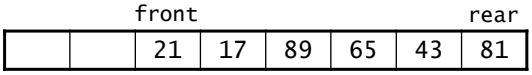
Avoiding the Need to Shift Items

- Problem: what do we do when we reach the end of the array?

  example: a queue of integers:



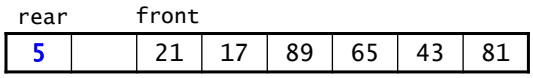
  the same queue after removing two items and inserting two:



  we have room for more items, but shifting to make room is inefficient

- Solution: maintain a *circular queue*. When we reach the end of the array, we wrap around to the beginning.

  insert 5: wrap around!

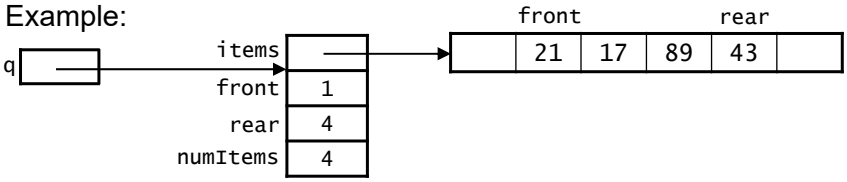


Maintaining a Circular Queue

- We use the mod operator (%) when updating `front` or `rear`:

```
front = (front + 1) % items.length;
rear = (rear + 1) % items.length;
```

- Example:



Maintaining a Circular Queue

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```
front = (front + 1) % items.length;
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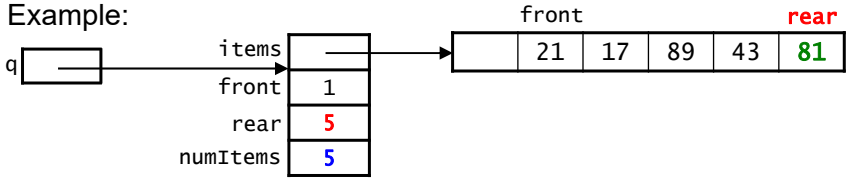
- Example:



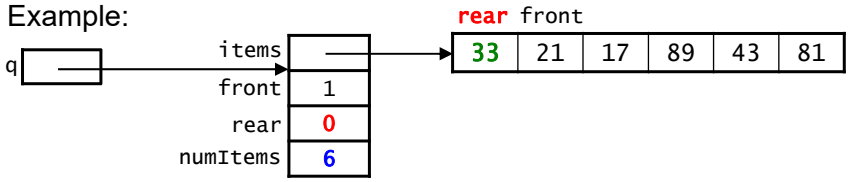
- `q.insert(81):` *// rear is not at end of array*
  - `rear = (rear + 1) % items.length;`
    `= ( 4 + 1) % 6`
    `= 5 % 6 = 5` *(% has no effect)*

Maintaining a Circular Queue

- We use the mod operator (%) when updating `front` or `rear`:

```
front = (front + 1) % items.length;
rear = (rear + 1) % items.length;
```

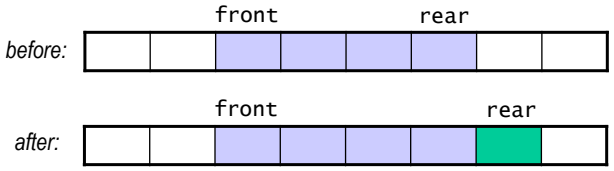
- Example:



- `q.insert(81):` *// rear is not at end of array*
  - `rear = (rear + 1) % items.length;`
    `= ( 4 + 1) % 6`
    `= 5 % 6 = 5` *(% has no effect)*
- `q.insert(33):` *// rear is at end of array*
  - `rear = (rear + 1) % items.length;`
    `= ( 5 + 1) % 6`
    `= 6 % 6 = 0` ***wrap around!***

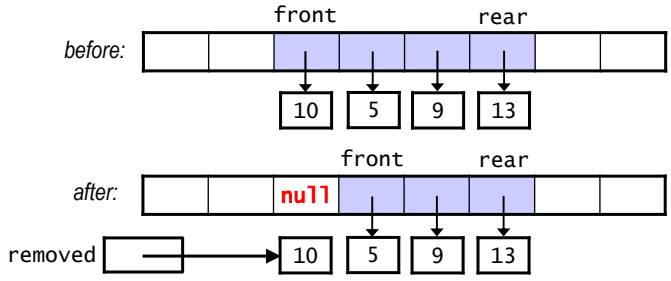
Inserting an Item in an ArrayQueue

- We increment `rear` before adding the item:




```
public boolean insert(T item) {
    // code to check for a null item goes here
    if (isFull()) {
        return false;
    }
    rear = (rear + 1) % items.length;
    items[rear] = item;
    numItems++;
    return true;
}
```

ArrayQueue remove()

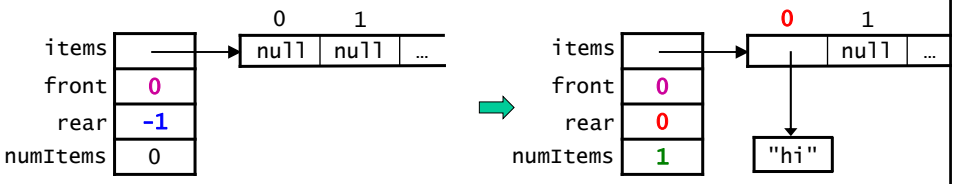



```
public T remove() {
    if (isEmpty()) {
        return null;
    }
    T removed = ____________________;


    numItems--;
    return removed;
}
```

Constructor

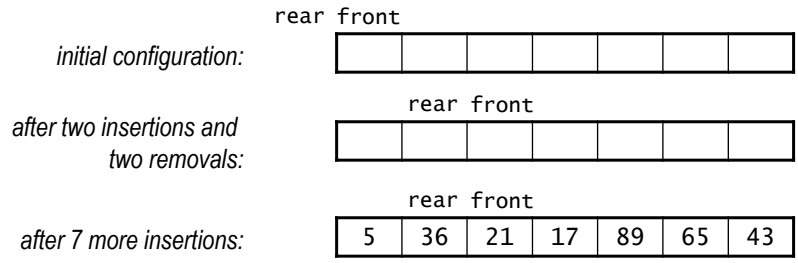
```
public ArrayQueue(int maxSize) {
    // code to check for an invalid maxSize goes here...
    items = (T[])new Object[maxSize];
    front = 0;
    rear = -1;
    numItems = 0;
}
```

- When we insert the first item in a newly created `ArrayQueue`, we want it to go in position 0. Thus, we need to:
  - start `rear` at `-1`, since then it will be incremented to `0` and used to perform the insertion
  - start `front` at `0`, since it is not changed by the insertion



Testing if an `ArrayQueue` is Empty or Full

- In both empty and full queues, `rear` is one "behind" `front`:



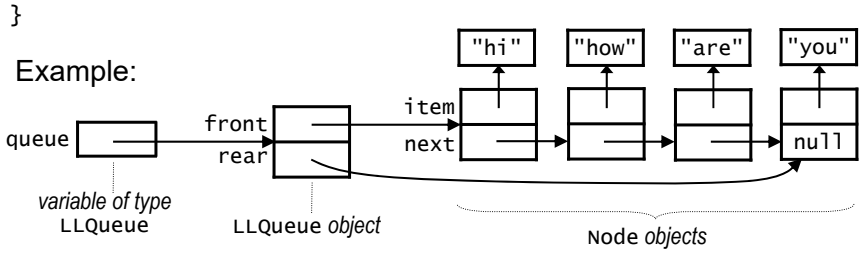
- This is why we maintain `numItems`!

```
public boolean isEmpty() {
    return (numItems == 0);
}

public boolean isFull() {
    return (numItems == items.length);
}
```

Implementing a Queue Using a Linked List

```
public class LLQueue<T> implements Queue<T> {
    private Node front;     // front of the queue
    private Node rear;      // rear of the queue
    …
}
```

- Example:

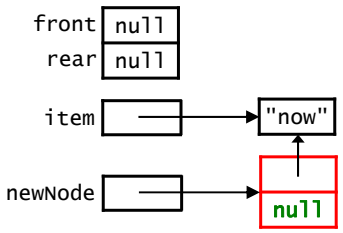


- In a linked list, we can efficiently:
  - remove the item at the front
  - add an item to the rear (if we have a ref. to the last node)
- Thus, this implementation is simpler than the array-based one!

Other Details of Our LLQueue Class

```
public class LLQueue<T> implements Queue<T> {
    private class Node {
        private T item;
        private Node next;
        ...
    }

    private Node front;
    private Node rear;

    public LLQueue() {
        front = null;
        rear = null;
    }
    public boolean isEmpty() {
        return (front == null);
    }
    public boolean isFull() {
        return false;
    }
    …
}
```

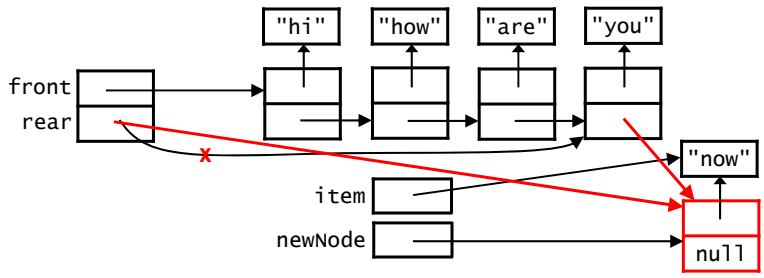
Inserting an Item in an Empty LLQueue



The `next` *field in the* `newNode` *will be* `null` *regardless of whether the queue is empty. Why?*

```
public boolean insert(T item) {
    // code to check for a null item goes here
    Node newNode = new Node(item, null);
    if (isEmpty()) {
        front = newNode;
        rear = newNode;
    } else {
        // we'll add this later!

    }
    return true;
}
```

Inserting an Item in a Non-Empty LLQueue




```
public boolean insert(T item) {
    // code to check for a null item goes here
    Node newNode = new Node(item, null);
    if (isEmpty()) {
        front = newNode;
        rear = newNode;
    } else {


    }
    return true;
}
```

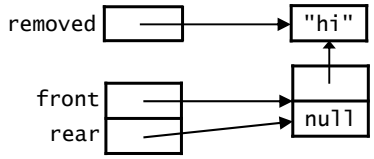
A. `rear = newNode;`
   `rear.next = newNode;`

B. `rear.next = newNode;`
   `rear = newNode;`

C. either A or B

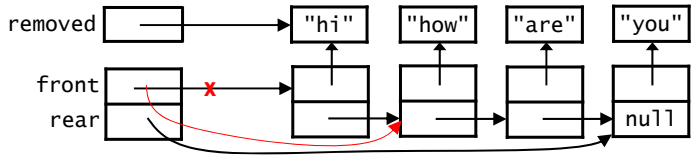
D. neither A nor B

Removing from an LLQueue with One Item




```
public T remove() {
    if (isEmpty()) {
        return null;
    }

    T removed = ____________________;
    if (front == rear) {     // removing the only item
        front = null;
        rear = null;
    } else {
        // we'll add this later
    }

    return removed;
}
```

Removing from an LLQueue with Two or More Items




```
public T remove() {
    if (isEmpty()) {
        return null;
    }

    T removed = ____________________;
    if (front == rear) {     // removing the only item
        front = null;
        rear = null;
    } else {

    }

    return removed;
}
```

Efficiency of the Queue Implementations

	`ArrayQueue`	`LLQueue`
`insert()`	*O*(1)	*O*(1)
`remove()`	*O*(1)	*O*(1)
`peek()`	*O*(1)	*O*(1)
space efficiency	*O*(m) where m is the *anticipated* maximum number of items	*O*(n) where n is the number of items currently in the queue

Applications of Queues

- first-in first-out (FIFO) inventory control
- OS scheduling: processes, print jobs, packets, etc.
- simulations of banks, supermarkets, airports, etc.

Binary Trees and Huffman Encoding

Computer Science E-22
Harvard University

David G. Sullivan, Ph.D.

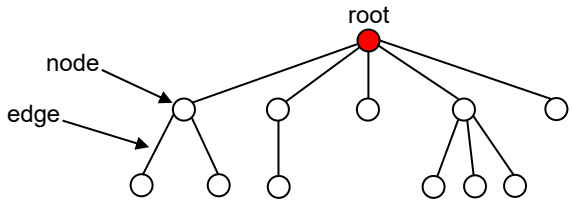
Motivation: Implementing a Dictionary

- A *data dictionary* is a collection of data with two main operations:
  - *search* for an item (and possibly delete it)
  - *insert* a new item
- If we use a *sorted* list to implement it, efficiency = $O(n)$.

data structure	***searching for an item***	***inserting an item***
a list implemented using an array	$O(\log n)$ using binary search	$O(n)$ because we need to shift items over
a list implemented using a linked list	$O(n)$ using linear search (binary search in a linked list is $O(n \log n)$)	$O(n)$ ($O(1)$ to do the actual insertion, but $O(n)$ to find where it belongs)

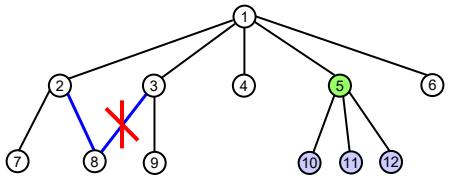
- In the next few lectures, we'll look at how we can use a *tree* for a data dictionary, and we'll try to get better efficiency.
- We'll also look at other applications of trees.

What Is a Tree?



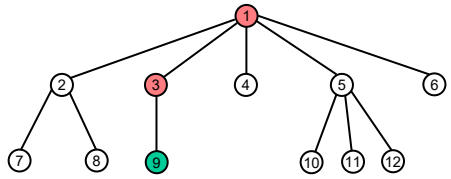
- A tree consists of:
  - a set of *nodes*
  - a set of *edges*, each of which connects a pair of nodes
- Each node may have one or more *data items.*
  - each data item consists of one or more fields
  - *key field* = the field used when searching for a data item
  - data items with the same key are referred to as *duplicates*
- The node at the "top" of the tree is called the *root* of the tree.

Relationships Between Nodes



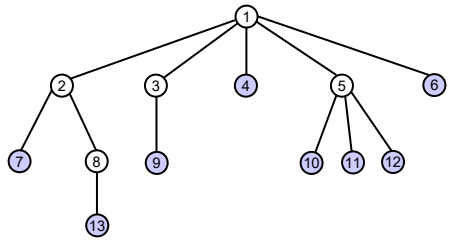
- If a node N is connected to nodes directly below it in the tree:
  - N is referred to as their *parent*
  - they are referred to as its *children*.
    - example: node 5 is the parent of nodes 10, 11, and 12
- Each node is the child of *at most one* parent.
- Nodes with the same parent are *siblings.*

Relationships Between Nodes (cont.)



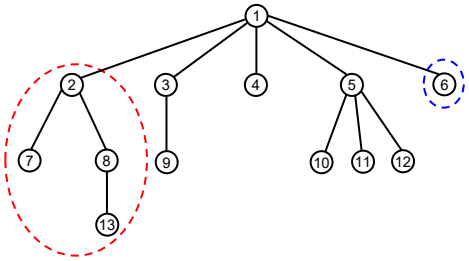
- A node's *ancestors* are its parent, its parent's parent, etc.
  - example: node 9's ancestors are 3 and 1
- A node's *descendants* are its children, their children, etc.
  - example: node 1's descendants are *all* of the other nodes

Types of Nodes



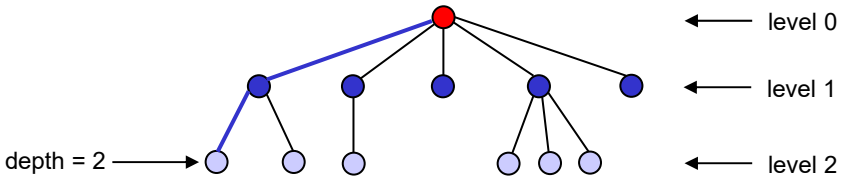
- A *leaf node* is a node without children.
- An *interior node* is a node with one or more children.

A Tree is a Recursive Data Structure



- Each node in the tree is the root of a smaller tree!
  - refer to such trees as *subtrees* to distinguish them from the tree as a whole
  - example: node 2 is the root of the subtree circled above
  - example: node 6 is the root of a subtree with only one node
- We'll see that tree algorithms often lend themselves to recursive implementations.

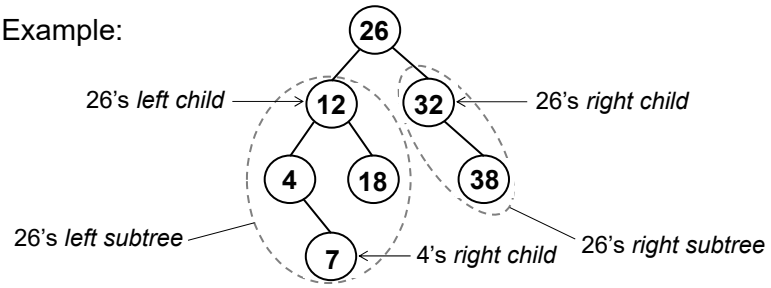
Path, Depth, Level, and Height



- There is exactly one *path* (one sequence of edges) connecting each node to the root.
- *depth* of a node = # of edges on the path from it to the root
- Nodes with the same depth form a *level* of the tree.
- The *height* of a tree is the maximum depth of its nodes.
  - example: the tree above has a height of 2

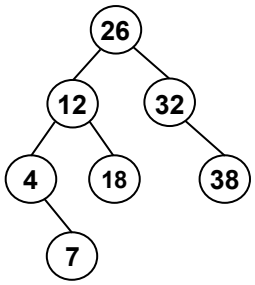
Binary Trees

- In a *binary tree*, nodes have *at most two* children.
  - distinguish between them using the direction *left* or *right*
- Example:



- Recursive definition: a binary tree is either:
  1) empty, or
  2) a node (the root of the tree) that has:
     - one or more pieces of data (the key, and possibly others)
     - a *left subtree*, which is itself a binary tree
     - a *right subtree*, which is itself a binary tree

Which of the following is/are <u>not</u> true?



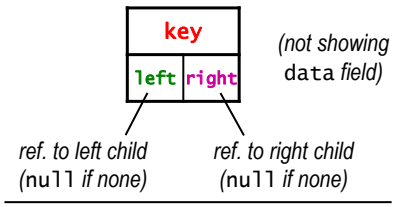
A. This tree has a height of 4.
B. There are 3 leaf nodes.
C. The 38 node is the right child of the 32 node.
D. The 12 node has 3 children.
E. more than one of the above are <u>not</u> true (which ones?)

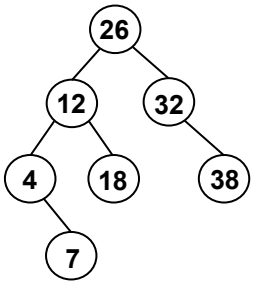
Representing a Binary Tree Using Linked Nodes

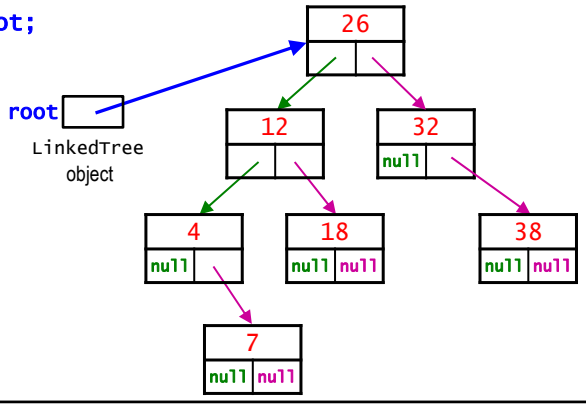
```
public class LinkedTree {
    private class Node {
        private int key;       // limit ourselves to int keys
        private LLList data;   // list of data for that key
        private Node left;     // reference to left child
        private Node right;    // reference to right child
        …
    }

    private Node root;
    …
}
```

Representing a Binary Tree Using Linked Nodes

```
public class LinkedTree {
    private class Node {
        private int key;
        private LLList data;
        private Node left;
        private Node right;
        …
    }

    private Node root;
    …
}
```

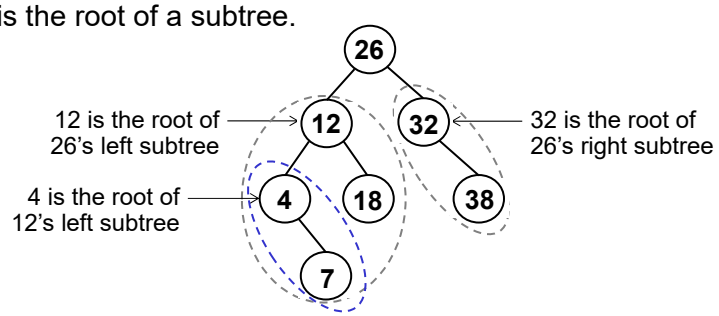






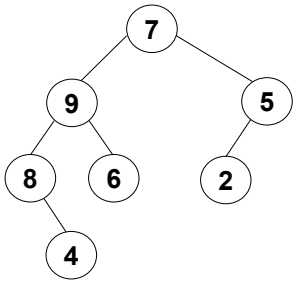
Traversing a Binary Tree

- Traversing a tree involves *visiting* all of the nodes in the tree.
  - visiting a node = processing its data in some way
    - example: print the key
- We'll look at four types of traversals.
  - each visits the nodes in a different order
- To understand traversals, it helps to remember that every node is the root of a subtree.



1: Preorder Traversal

- preorder traversal of the tree whose root is N:
  1) visit the root, N
  2) recursively perform a preorder traversal of N's left subtree
  3) recursively perform a preorder traversal of N's right subtree



- *pre*order because a node is visited *before* its subtrees
- The root of the tree as a whole is visited first.

Implementing Preorder Traversal

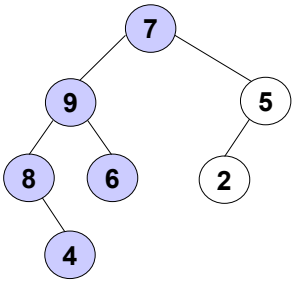
```
public class LinkedTree {
    ...
    private Node root;

    public void preorderPrint() {
        if (root != null) {
            preorderPrintTree(root);
        }
        System.out.println();
    }
    private static void preorderPrintTree(Node root) {
        System.out.print(root.key + " ");
        if (root.left != null) {
            preorderPrintTree(root.left);
        }
        if (root.right != null) {
            preorderPrintTree(root.right);
        }
    }
```

Not always the same as the root of the entire tree.

- `preorderPrintTree()` is a static, recursive method that takes the root of the tree/subtree that you want to print.
- `preorderPrint()` is a non-static "wrapper" method that makes the initial call. It passes in the root of the entire tree.

Tracing Preorder Traversal

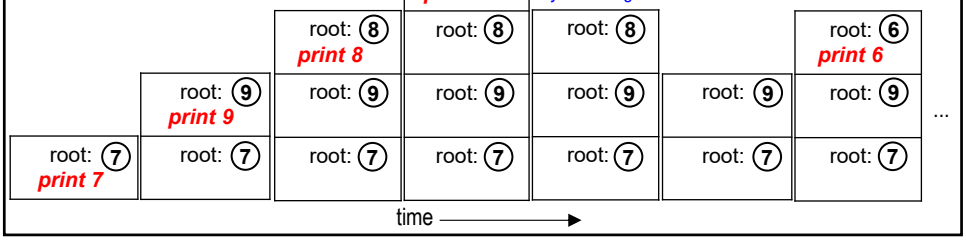
```
void preorderPrintTree(Node root) {
   System.out.print(root.key + " ");
   if (root.left != null) {
       preorderPrintTree(root.left);
   }
   if (root.right != null) {
       preorderPrintTree(root.right);
   }
}
```



order in which nodes are visited:

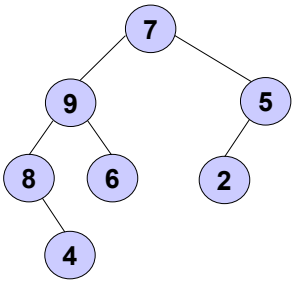
base case, since neither recursive call is made!

we go back up the tree by returning!



Using Recursion for Traversals

```
void preorderPrintTree(Node root) {
   System.out.print(root.key + " ");
   if (root.left != null) {
       preorderPrintTree(root.left);
   }
   if (root.right != null) {
       preorderPrintTree(root.right);
   }
}
```



order in which nodes are visited:

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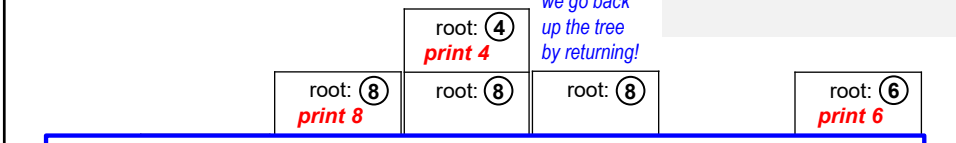
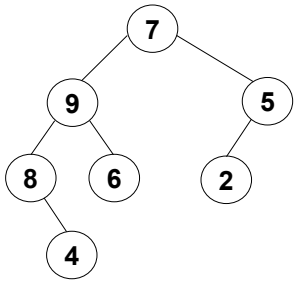


- Using recursion allows us to easily go back up the tree.
- Using a loop would be harder. Why?

...

2: Postorder Traversal

- postorder traversal of the tree whose root is N:
  1) recursively perform a postorder traversal of N's left subtree
  2) recursively perform a postorder traversal of N's right subtree
  3) visit the root, N



- *post*order because a node is visited *after* its subtrees
- The root of the tree as a whole is visited last.

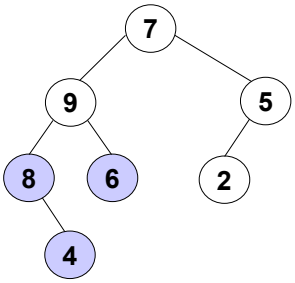
Implementing Postorder Traversal

```
public class LinkedTree {
    …
    private Node root;

    public void postorderPrint() {
        if (root != null) {
            postorderPrintTree(root);
        }
        System.out.println();
    }

    private static void postorderPrintTree(Node root) {
        if (root.left != null) {
            postorderPrintTree(root.left);
        }
        if (root.right != null) {
            postorderPrintTree(root.right);
        }
        System.out.print(root.key + " ");
    }
```

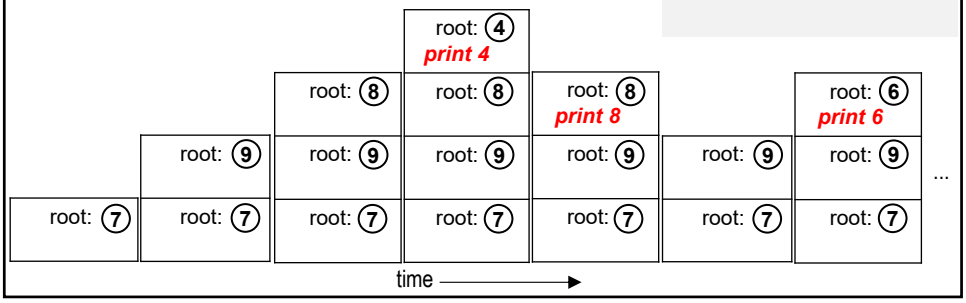
- Note that the root is printed *after* the two recursive calls.

Tracing Postorder Traversal

```
void postorderPrintTree(Node root) {
   if (root.left != null) {
       postorderPrintTree(root.left);
   }
   if (root.right != null) {
       postorderPrintTree(root.right);
   }
   System.out.print(root.key + " ");
}
```

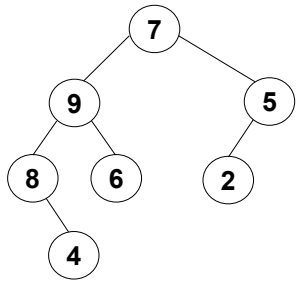


order in which nodes are visited:



3: Inorder Traversal

- inorder traversal of the tree whose root is N:
  1) recursively perform an inorder traversal of N's left subtree
  2) visit the root, N
  3) recursively perform an inorder traversal of N's right subtree



- The root of the tree as a whole is visited between its subtrees.
- We'll see later why this is called *in*order traversal!

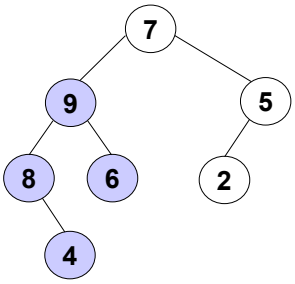
Implementing Inorder Traversal

```
public class LinkedTree {
    …
    private Node root;

    public void inorderPrint() {
        if (root != null) {
            inorderPrintTree(root);
        }
        System.out.println();
    }

    private static void inorderPrintTree(Node root) {
        if (root.left != null) {
            inorderPrintTree(root.left);
        }
        System.out.print(root.key + " ");
        if (root.right != null) {
            inorderPrintTree(root.right);
        }
    }
}
```

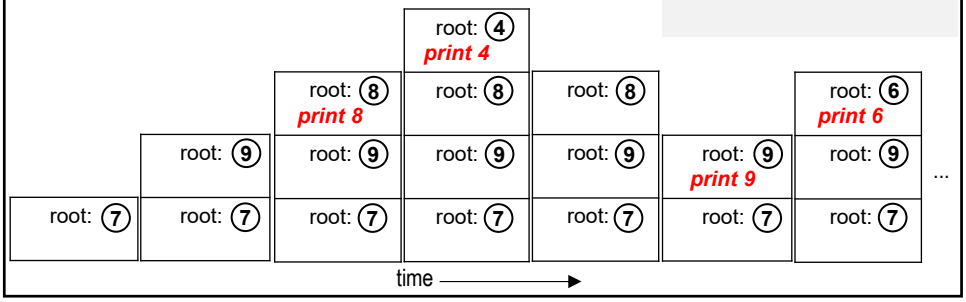
- Note that the root is printed *between* the two recursive calls.

Tracing Inorder Traversal

```
void inorderPrintTree(Node root) {
   if (root.left != null) {
       inorderPrintTree(root.left);
   }
   System.out.print(root.key + " ");
   if (root.right != null) {
       inorderPrintTree(root.right);
   }
}
```

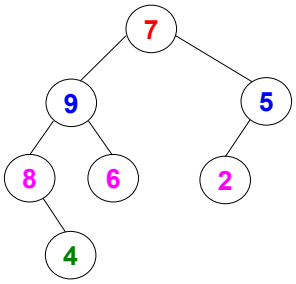


order in which nodes are visited:



Level-Order Traversal

- Visit the nodes one level at a time, from top to bottom and left to right.

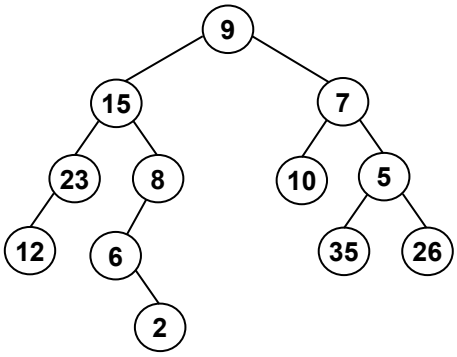


- Level-order traversal of the tree above: 7 9 5 8 6 2 4
- We can implement this type of traversal using a queue.

Tree-Traversal Summary

preorder: root, left subtree, right subtree
postorder: left subtree, right subtree, root
inorder: left subtree, root, right subtree
level-order: top to bottom, left to right

- Perform each type of traversal on the tree below:



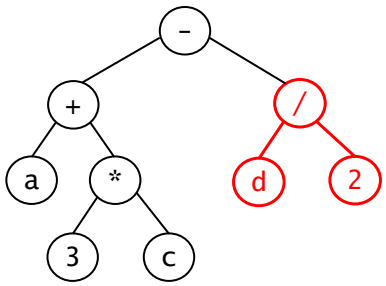
Tree Traversal Puzzle

- preorder traversal: A M P K L D H T
- inorder traversal: P M L K A H T D
- Draw the tree!

- What's one fact that we can easily determine from one of the traversals?

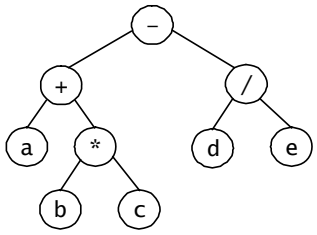
Using a Binary Tree for an Algebraic Expression

- We'll restrict ourselves to fully parenthesized expressions using the following binary operators: `+`, –, `*`, `/`
- Example: `((a + (3 * c)) - (d / 2))`



- Leaf nodes are variables or constants.
- Interior nodes are operators.
  - their children are their operands

Traversing an Algebraic-Expression Tree



- Inorder gives conventional algebraic notation.
  - print '(' before the recursive call on the left subtree
  - print ')' after the recursive call on the right subtree
  - for tree at right: `((a + (b * c)) - (d / e))`
- Preorder gives functional notation.
  - print '('s and ')'s as for inorder, and commas after the recursive call on the left subtree
  - for tree above: `subtr(add(a, mult(b, c)), divide(d, e))`
- Postorder gives the order in which the computation must be carried out on a stack/RPN calculator.
  - for tree above: `push a, push b, push c, multiply, add,…`

Fixed-Length Character Encodings

- A character encoding maps each character to a number.
- Computers usually use fixed-length character encodings.
  - ASCII - 8 bits per character

char	dec	binary
'a'	97	01100001
'b'	98	01100010
…	…	…
't'	116	01110100

example: "bat" is stored in a text file as the following sequence of bits:

01100010 01100001 01110100

  - Unicode - 16 bits per character
    (allows for foreign-language characters; ASCII is a subset)
- Fixed-length encodings are simple, because:
  - all encodings have the same length
  - a given character always has the same encoding

A Problem with Fixed-Length Encodings

- They tend to waste space.
- Example: an English newspaper article with only:
  - upper and lower-case letters (52 characters)
  - spaces and newlines (2 characters)
  - common punctuation (approx. 10 characters)
  - total of 64 unique characters ➔ only need ___ bits
- We could gain even more space if we:
  - gave the most common letters shorter encodings (3 or 4 bits)
  - gave less frequent letters longer encodings (> 6 bits)

Variable-Length Character Encodings

- Variable-length encodings *compress* a text file by:
  - using encodings of different lengths for different characters
  - assigning shorter encodings to frequently occurring characters
- Example: if we had only four characters

e	01
o	100
s	111
t	00

"test" would be encoded as
00 01 111 00 → 000111100

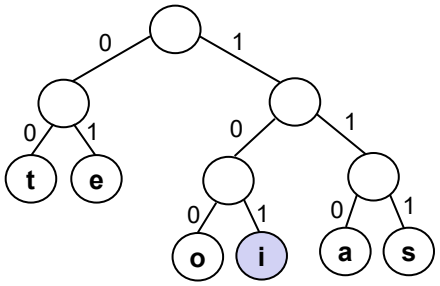
- Challenge: when reading a document, how do we determine the boundaries between characters?
  - how do we know how many bits the next character has?
- One requirement: no character's encoding can be the prefix of another character's encoding (e.g., couldn't have 00 and 001).

Huffman Encoding

- One type of variable-length encoding
- Based on the actual character frequencies in a given document
  - different documents have different encodings
- Huffman encoding uses a binary tree:
  - to determine the encoding of each character
  - to *decode / decompress* an encoded file
    - putting it back into ASCII

Huffman Trees

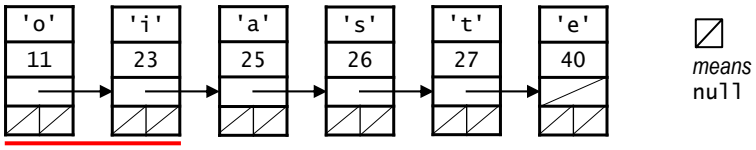
- Example for a text with only six characters:



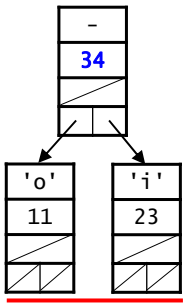
- Left branches are labeled with a 0, right branches with a 1.
- Leaf nodes are characters.
- To get a character's encoding, follow the path from the root to its leaf node.
  - example: **i** = ?

Building a Huffman Tree

1) Begin by reading through the text to determine the frequencies.
2) Create a list of nodes containing (character, frequency) pairs for each character in the text – *sorted by frequency.*

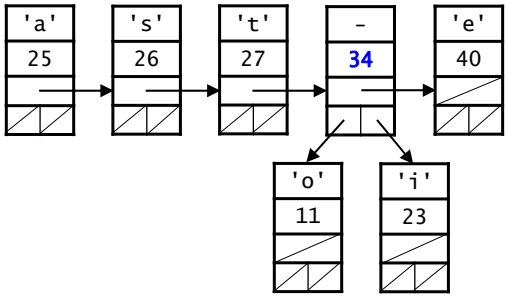


3) Remove and "merge" the nodes with the two lowest frequencies, forming a new node that is their parent.
   - left child = lowest frequency node
   - right child = the other node
   - frequency of parent = sum of the frequencies of its children
     - in this case, 11 + 23 = 34



Building a Huffman Tree (cont.)

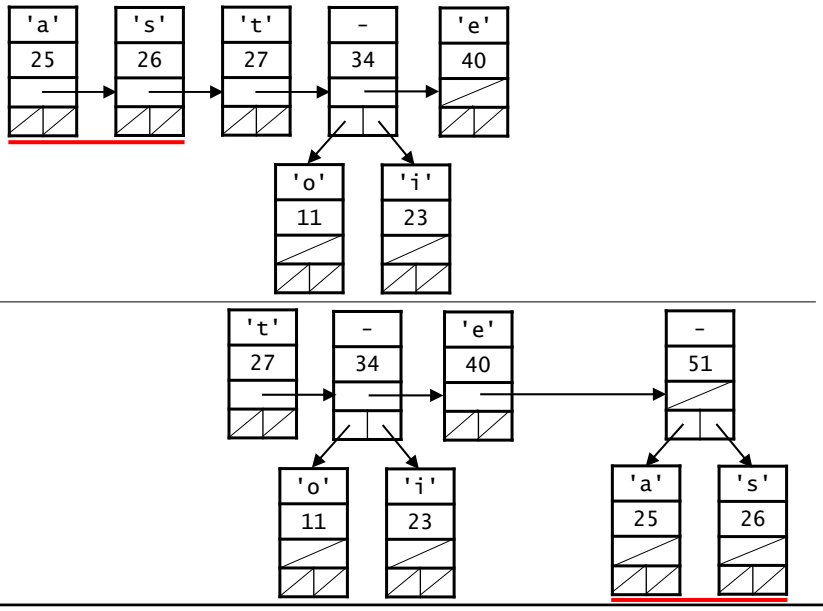
4) Add the parent to the list of nodes (maintaining sorted order):



5) Repeat steps 3 and 4 until there is only a single node in the list, which will be the root of the Huffman tree.

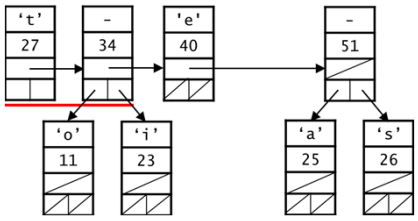
Completing the Huffman Tree Example I

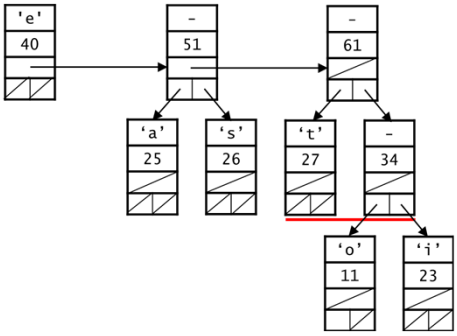
- Merge the two remaining nodes with the lowest frequencies:



Completing the Huffman Tree Example II

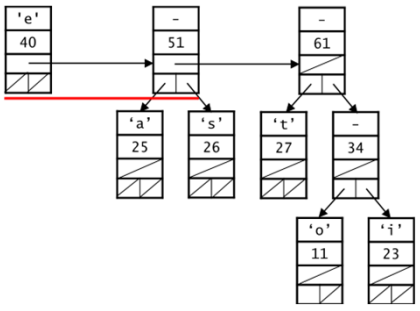
- Merge the next two nodes:

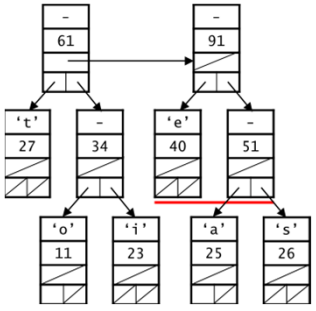




Completing the Huffman Tree Example II

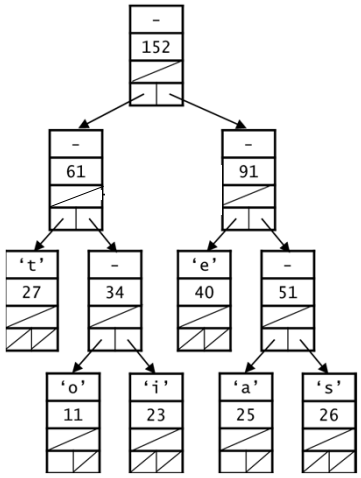
- Merge again:

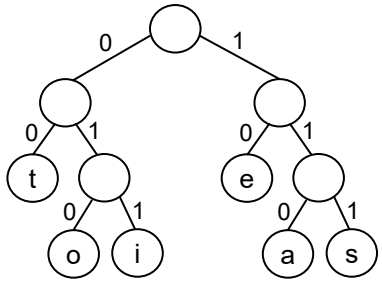




Completing the Huffman Tree Example IV

- The next merge creates the final tree:

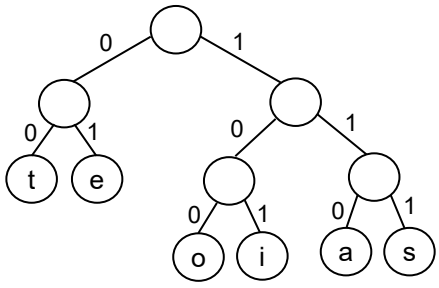




- Characters that appear more frequently end up higher in the tree, and thus their encodings are shorter.

The Shape of the Huffman Tree

- The tree on the last slide is fairly symmetric.
- This won't always be the case!
  - depends on the character frequencies
- For example, changing the frequency of `'o'` from 11 to 21 would produce the tree shown below:



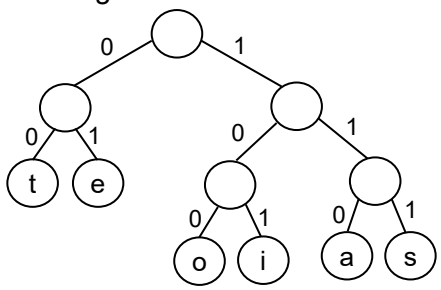
- This is the tree that we'll use in the remaining slides.

Huffman Encoding: Compressing a File

1) Read through the input file and build its Huffman tree.

2) Write a file header for the output file.
   - include the character frequencies so the tree can be rebuilt when the file is decompressed

3) Traverse the Huffman tree to create a table containing the encoding of each character:



a	
e	
i	
o	
s	
t	

4) Read through the input file a second time, and write the Huffman code for each character to the output file.

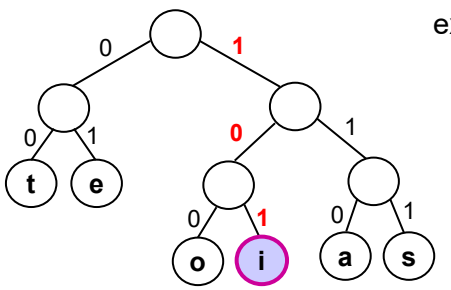
Huffman Decoding: Decompressing a File

1) Read the frequency table from the header and rebuild the tree.

2) Read one bit at a time and traverse the tree, starting from the root:

   when you read a bit of 1, go to the right child
   when you read a bit of 0, go to the left child
   when you reach a leaf node, record the character,
   return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!



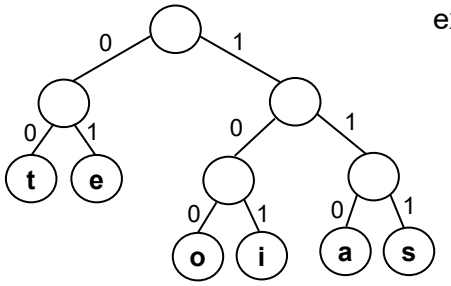
example: **101**111110000111100
first character = i

What are the next three characters?

1) Read the frequency table from the header and rebuild the tree.
2) Read one bit at a time and traverse the tree, starting from the root:

when you read a bit of 1, go to the right child
when you read a bit of 0, go to the left child
when you reach a leaf node, record the character,
return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!



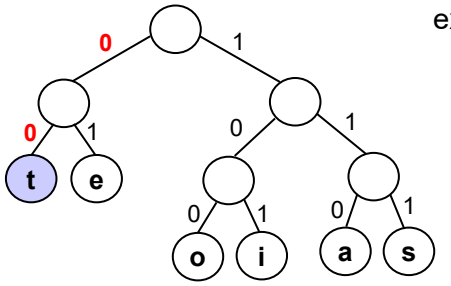
example: 101111110000111100
first character = i (101)

Huffman Decoding: Decompressing a File

1) Read the frequency table from the header and rebuild the tree.
2) Read one bit at a time and traverse the tree, starting from the root:

   when you read a bit of 1, go to the right child
   when you read a bit of 0, go to the left child
   when you reach a leaf node, record the character,
   return to the root, and continue reading bits

The tree allows us to easily overcome the challenge of determining the character boundaries!



example: 1011111100001111**00**

```
101 =  right,left,right   = i
111 =  right,right,right= s
110 =  right,right,left   = a
 00 =  left,left          = t
 01 =  left,right         = e
111 =  right,right,right= s
 00 =  left,left          = t
```

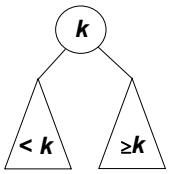
Search Trees

Computer Science E-22
Harvard University

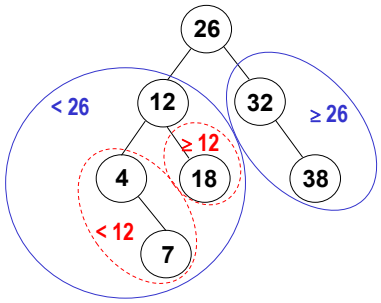
David G. Sullivan, Ph.D.

Binary *Search* Trees

- Search-tree property: for each node *k* (*k* is the key):
  - all nodes in *k*'s left subtree are < *k*
  - all nodes in *k*'s right subtree are >= *k*



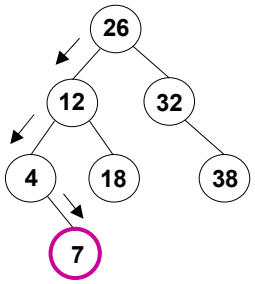
- Our earlier binary-tree example is a search tree:



- With a search tree, an inorder traversal visits the nodes in order!
  - in order of increasing key values

Searching for an Item in a Binary Search Tree

- Algorithm for searching for an item with a key *k*:
  if *k* == the root node's key, you're done
  else if *k* < the root node's key, search the left subtree
  else search the right subtree
- Example: search for 7



Implementing Binary-Tree Search

```
public class LinkedTree {   // Nodes have keys that are ints
    …
    private Node root;

    public LLList search(int key) {      // "wrapper method"
        Node n = searchTree(root, key);  // get Node for key
        if (n == null) {
            return null;     // no such key
        } else {
            return n.data;   // return list of values for key
        }
    }

    private static Node searchTree(Node root, int key) {
        if (                ) {
        } else if (             ) {
        } else if (             ) {
        } else {
        }
    }
```

two base cases (order matters!)

two recursive cases

Inserting an Item in a Binary Search Tree

- `public void insert(int key, Object data)`
  will add a new (key, data) pair to the tree
- Example 1: a search tree containing student records
  - key = the student's ID number (an integer)
  - data = a string with the rest of the student record
  - we want to be able to write client code that looks like this:

```
LinkedTree students = new LinkedTree();
students.insert(23, "Jill Jones,sophomore,comp sci");
students.insert(45, "Al Zhang,junior,english");
```

- Example 2: a search tree containing scrabble words
  - key = a scrabble score (an integer)
  - data = a word with that scrabble score

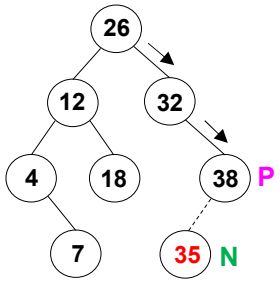
```
LinkedTree tree = new LinkedTree();
tree.insert(4, "lost");
```

Inserting an Item in a Binary Search Tree (cont.)

- To insert an item (*k*, *d*), we start by searching for *k*.
- If we find a node with key *k*, we add *d* to the list of data values for that node.
  - example: `tree.insert(4, "sail")`
- If we don't find *k*, the last node seen in the search becomes the parent **P** of the new node **N**.
  - if *k* < **P**'s key, make **N** the left child of **P**
  - else make **N** the right child of **P**
- *Special case:* if the tree is empty, make the new node the root of the tree.
- ***Important:*** The resulting tree is still a search tree!

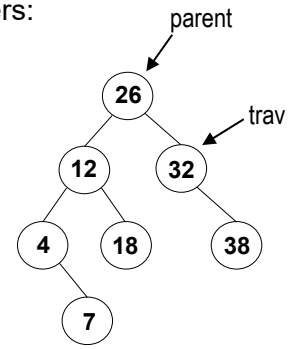
example:

```
tree.insert(35,
  "photooxidizes")
```



Implementing Binary-Tree Insertion

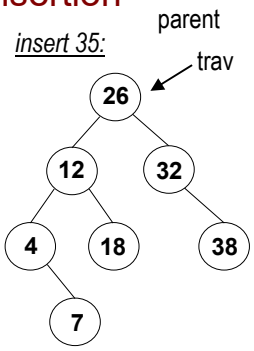
- We'll implement part of the `insert()` method together.
- We'll use iteration rather than recursion.
- Our method will use two references/pointers:
  - `trav`: performs the traversal down to the point of insertion
  - `parent`: stays one behind `trav`
    - like the `trail` reference that we sometimes use when traversing a linked list



Implementing Binary-Tree Insertion

```
public void insert(int key, Object data) {
    Node parent = null;
    Node trav = root;
    while (trav != null) {
        if (trav.key == key) {
            trav.data.addItem(data, 0);
            return;
        }
        // what should go here?
```

insert 35:

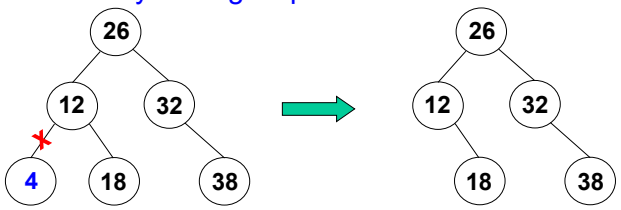



```
    }
    Node newNode = new Node(key, data);
    if (root == null) {   // the tree was empty
        root = newNode;
    } else if (key < parent.key) {
        parent.left = newNode;
    } else {
        parent.right = newNode;
    }
}
```

Deleting Items from a Binary Search Tree

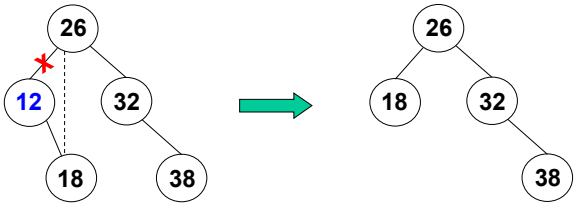
- Three cases for deleting a node *x*
- **Case 1:** *x* has no children.
  Remove *x* from the tree by setting its parent's reference to null.

ex: delete 4



- **Case 2:** *x* has one child.
  Take the parent's reference to *x* and make it refer to *x*'s child.

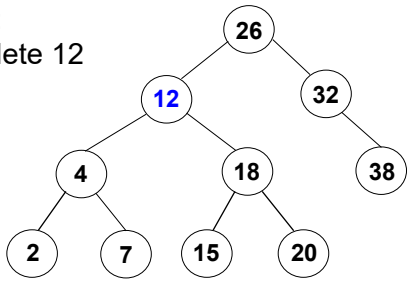
ex: delete 12



Deleting Items from a Binary Search Tree (cont.)

- **Case 3:** *x* has two children
  - we can't give both children to the parent. why?
  - instead, we leave *x's* node where it is, and we replace its key and data with those from another node
    - the replacement must maintain the search-tree inequalities

ex:
delete 12

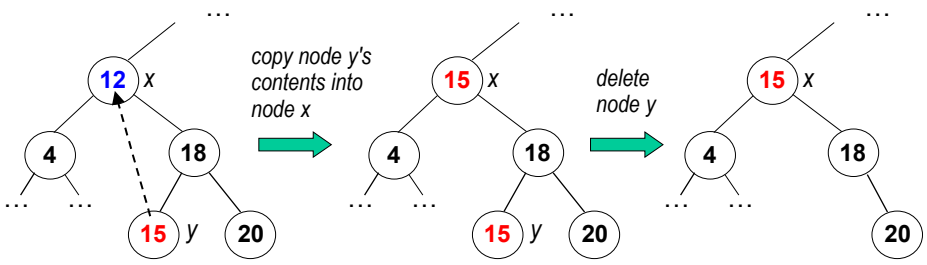


two options: *which ones?*

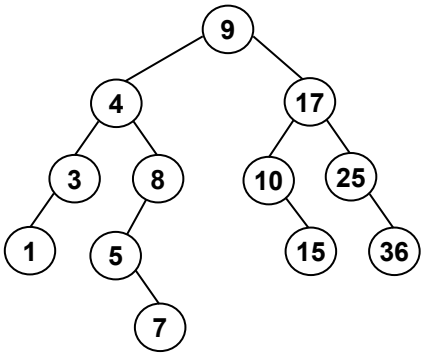
Deleting Items from a Binary Search Tree (cont.)

- **Case 3:** *x* has two children (continued):
  - replace *x*'s key and data with those from the smallest node in *x*'s right subtree—call it *y*
  - we then delete *y*
    - it will either be a leaf node or will have one right child. why?
    - thus, we can delete it using case 1 or 2

ex: delete 12

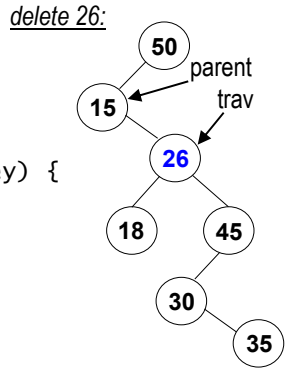


Which Node Would Be Used To Replace 9?



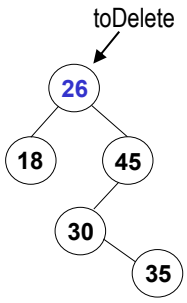
A. **4**
B. **8**
C. **10**
D. **15**
E. **17**

Implementing Deletion




```
public LLList delete(int key) {
    // Find the node and its parent.
    Node parent = null;
    Node trav = root;
    while (trav != null && trav.key != key) {
        parent = trav;
        if (key < trav.key) {
            trav = trav.left;
        } else {
            trav = trav.right;
        }
    }

    // Delete the node (if any) and return the removed items.
    if (trav == null) {   // no such key
        return null;
    } else {
        LLList removedData = trav.data;
        deleteNode(trav, parent);   // call helper method
        return removedData;
    }
}
```

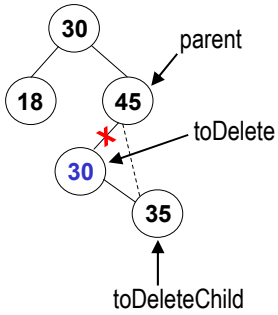
Implementing Case 3



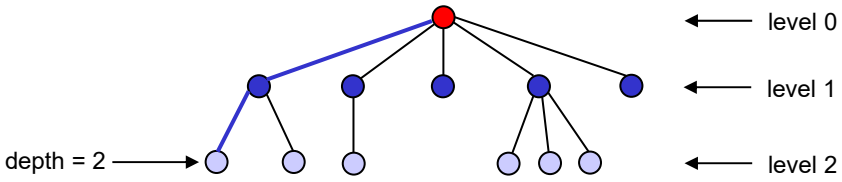

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        // Find a replacement - and
        // the replacement's parent.
        Node replaceParent = toDelete;

        // Get the smallest item
        // in the right subtree.
        Node replace = toDelete.right;
        // what should go here?




        // Replace toDelete's key and data
        // with those of the replacement item.
        toDelete.key = replace.key;
        toDelete.data = replace.data;

        // Recursively delete the replacement
        // item's old node. It has at most one
        // child, so we don't have to
        // worry about infinite recursion.
        deleteNode(replace, replaceParent);
    } else {
        ...
}
```

Implementing Cases 1 and 2

```
private void deleteNode(Node toDelete, Node parent) {
    if (toDelete.left != null && toDelete.right != null) {
        ...
    } else {
        Node toDeleteChild;
        if (toDelete.left != null)
            toDeleteChild = toDelete.left;
        else
            toDeleteChild = toDelete.right;
        // Note: in case 1, toDeleteChild
        // will have a value of null.

        if (toDelete == root)
            root = toDeleteChild;
        else if (toDelete.key < parent.key)
            parent.left = toDeleteChild;
        else
            parent.right = toDeleteChild;
    }
}
```



Recall: Path, Depth, Level, and Height



- There is exactly one *path* (one sequence of edges) connecting each node to the root.
- *depth* of a node = # of edges on the path from it to the root
- Nodes with the same depth form a *level* of the tree.
- The *height* of a tree is the maximum depth of its nodes.
    - example: the tree above has a height of 2

Efficiency of a Binary Search Tree

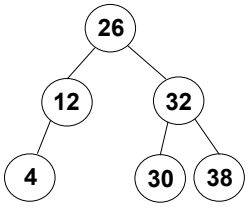
- For a tree containing *n* items, what is the efficiency of any of the traversal algorithms?
  - you process all *n* of the nodes
  - you perform *O*(1) operations on each of them
- Search, insert, and delete all have the same time complexity.
  - insert is a search followed by *O*(1) operations
  - delete involves either:
    - a search followed by *O*(1) operations (cases 1 and 2)
    - a search partway down the tree for the item, followed by a search further down for its replacement, followed by *O*(1) operations (case 3)

Efficiency of a Binary Search Tree (cont.)

- Time complexity of searching:
  - best case:
  - worst case:
    - you have to go all the way down to level *h* before finding the key or realizing it isn't there
    - along the path to level *h*, you process *h* + 1 nodes
  - average case:
- What is the height of a tree containing *n* items?

Balanced Trees

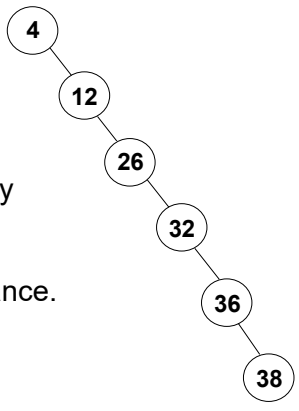
- A tree is *balanced* if, for *each* of its nodes, the node's subtrees have the same height or have heights that differ by 1.
  - example:
    - 26: both subtrees have a height of 1
    - 12: left subtree has height 0
      right subtree is empty (height = -1)
    - 32: both subtrees have a height of 0
    - all leaf nodes: both subtrees are empty



- For a balanced tree with n nodes, height = $O(\log n)$
  - each time that you follow an edge down the longest path, you cut the problem size roughly in half!
- Therefore, for a *balanced* binary search tree, the worst case for search / insert / delete is $O(h) = O(\log n)$
  - the "best" worst-case time complexity

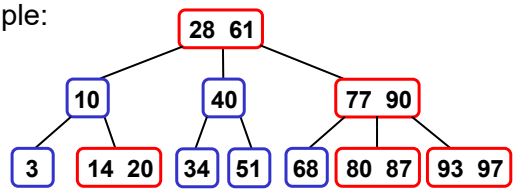
What If the Tree Isn't Balanced?

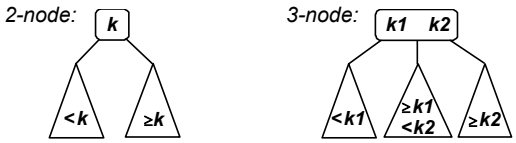
- Extreme case: the tree is equivalent to a linked list
  - height = $n - 1$
- Therefore, for a unbalanced binary search tree, the worst case for search / insert / delete is $O(h) = O(n)$
  - the "worst" worst-case time complexity
- We'll look next at search-tree variants that take special measures to ensure balance.



2-3 Trees

- A 2-3 tree is a balanced tree in which:
  - *all* nodes have equal-height subtrees (perfect balance)
  - each node is either
    - a **2-node**, which contains one data item and 0 or 2 children
    - a **3-node**, which contains two data items and 0 or 3 children
  - the keys form a search tree
- Example:

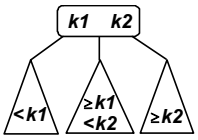




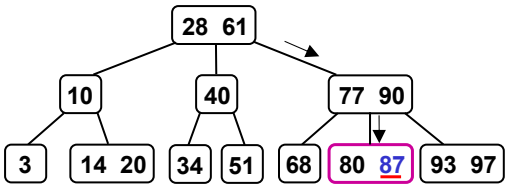
Search in 2-3 Trees

- Algorithm for searching for an item with a key *k*:

  if *k* == one of the root node's keys, you're done
  else if *k* < the root node's first key
  search the left subtree
  else if the root is a 3-node and k < its second key
  search the middle subtree
  else
  search the right subtree

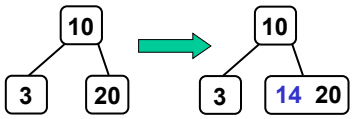


- Example: search for 87



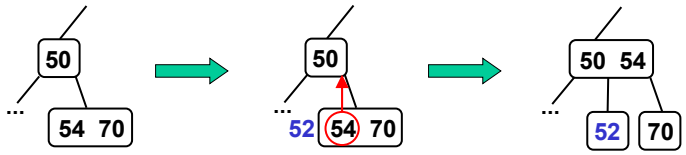
Insertion in 2-3 Trees

- Algorithm for inserting an item with a key *k*:
  search for *k,* but don't stop until you hit a leaf node
  let L be the leaf node at the end of the search
  if L is a 2-node
    add *k* to L, making it a 3-node



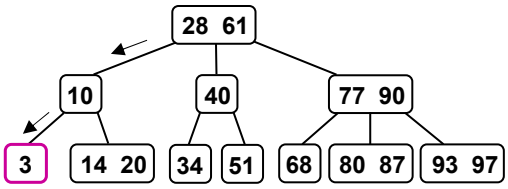
  else if L is a 3-node
    split L into two 2-nodes containing the items with the smallest and largest of: *k,* L's 1st key, L's 2nd key
    the middle item is "sent up" and inserted in L's parent

example: add 52

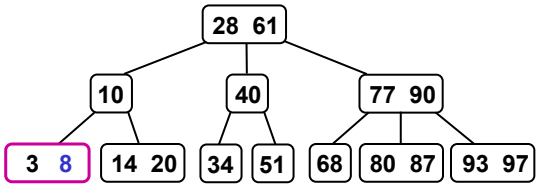


Example 1: Insert 8

- Search for 8:

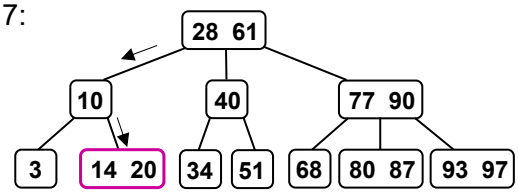


- Add 8 to the leaf node, making it a 3-node:

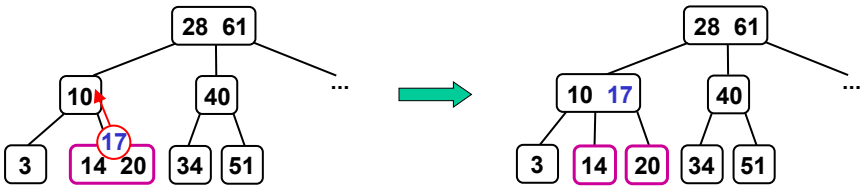


Example 2: Insert 17

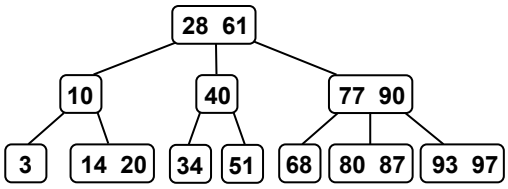
- Search for 17:



- Split the leaf node, and send up the middle of 14, 17, 20 and insert it the leaf node's parent:



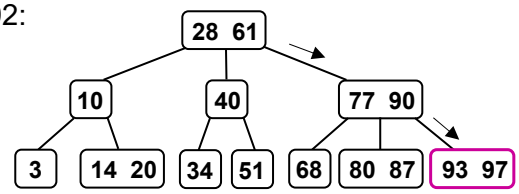
Example 3: Insert 92



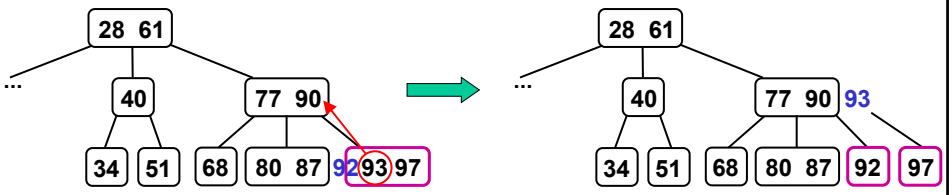
- In which node will we initially try to insert it?

Example 3: Insert 92

- Search for 92:



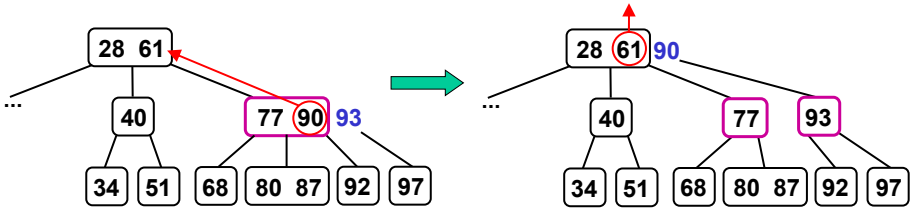
- Split the leaf node, and send up the middle of 92, 93, 97 and insert it the leaf node's parent:



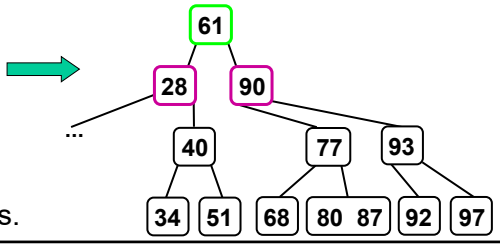
- In this case, the leaf node's parent is also a 3-node, so we need to split is as well…

Example 3 (cont.)

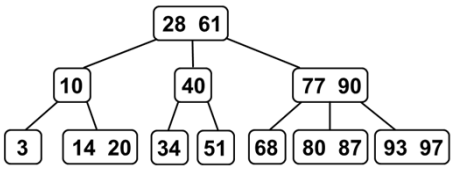
- We split the [77 90] node and we send up the middle of 77, 90, 93:
- We try to insert it in the root node, but the root is also full!



- Then we split the root, which increases the tree's height by 1, but the tree is still balanced.
- This is only case in which the tree's height increases.



Efficiency of 2-3 Trees



- A 2-3 tree containing n items has a height h <= $\log_2 n$.
- Thus, search and insertion are both *O*(log n).
  - search visits at most h + 1 nodes
  - insertion visits at most 2h + 1 nodes:
    - starts by going down the full height
    - in the worst case, performs splits all the way back up to the root
- Deletion is tricky – you may need to coalesce nodes!
  However, it also has a time complexity of *O*(log n).
- Thus, we can use 2-3 trees for a *O*(log n)-time data dictionary!

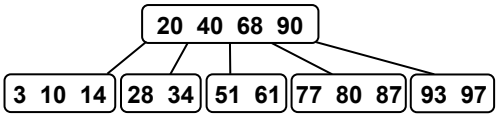
External Storage

- The balanced trees that we've covered don't work well if you want to store the data dictionary externally – i.e., on disk.
- Key facts about disks:
  - data is transferred to and from disk in units called *blocks,* which are typically 4 or 8 KB in size
  - disk accesses are slow!
    - reading a block takes ~10 milliseconds (10^{-3} sec)
    - vs. reading from memory, which takes ~10 nanoseconds
    - in 10 ms, a modern CPU can perform millions of operations!

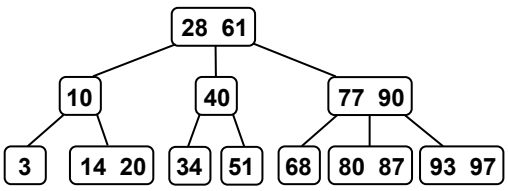
B-Trees

- A B-tree of order *m* is a tree in which each node has:
  - at most 2*m* entries (and, for internal nodes, 2*m* + 1 children)
  - at least *m* entries (and, for internal nodes, *m* + 1 children)
  - exception: the root node may have as few as 1 entry
  - a 2-3 tree is essentially a B-tree of order 1
- To minimize the number of disk accesses, we make *m* as large as possible.
  - each disk read brings in more items
  - the tree will be shorter (each level has more nodes), and thus searching for an item requires fewer disk reads
- A large value of *m* doesn't make sense for a memory-only tree, because it leads to many key comparisons per node.
- These comparisons are less expensive than accessing the disk, so large values of *m* make sense for on-disk trees.

Example: a B-Tree of Order 2



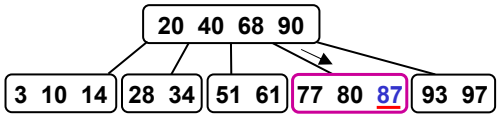
- *m* = 2: at most 2*m* = 4 items per node (and at most 5 children)
  at least m = 2 items per node (and at least 3 children)
  (except the root, which could have 1 item)
- The above tree holds the same keys this 2-3 tree:



- We used the same order of insertion to create both trees:
  51, 3, 40, 77, 20, 10, 34, 28, 61, 80, 68, 93, 90, 97, 87, 14

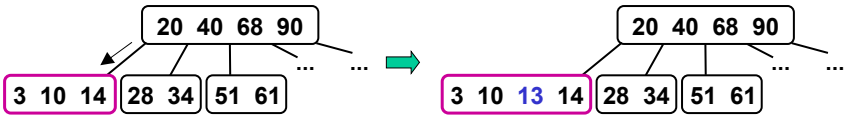
Search in B-Trees

- Similar to search in a 2-3 tree.
- Example: search for 87



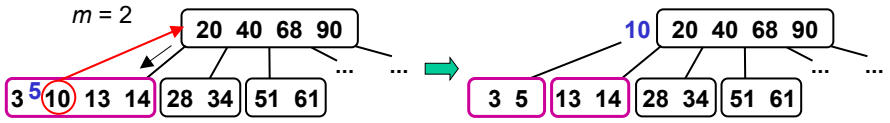
Insertion in B-Trees

- Similar to insertion in a 2-3 tree:
  - search for the key until you reach a leaf node
  - if a leaf node has fewer than 2*m* items, add the item to the leaf node
  - else split the node, dividing up the 2*m* + 1 items:
    - the smallest *m* items remain in the original node
    - the largest *m* items go in a new node
    - send the middle entry up and insert it (and a pointer to the new node) in the parent
- Example of an insertion without a split: insert 13

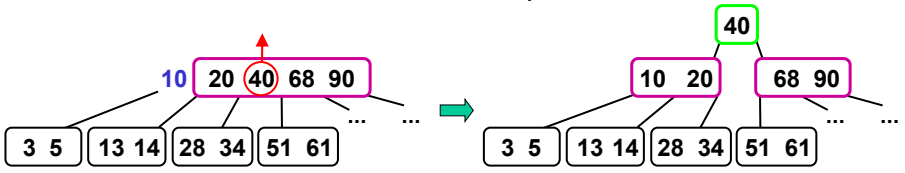


Splits in B-Trees

- Insert 5 into the result of the previous insertion:

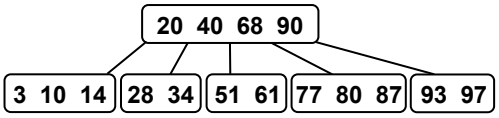


- The middle item (the 10) is sent up to the root.
  The root has no room, so it is also split, and a new root is formed:



- Splitting the root increases the tree's height by 1, but the tree is still balanced. This is only way that the tree's height increases.
- When an internal node is split, its $2m + 2$ pointers are split evenly between the original node and the new node.

Analysis of B-Trees



- All internal nodes have at least m children (actually, at least m+1).
- Thus, a B-tree with n items has a height <= $\log_m n$, and search and insertion are both $O(\log_m n)$.
- As with 2-3 trees, deletion is tricky, but it's still logarithmic.

Search Trees: Conclusions

- Binary search trees can be $O(\log n)$, but they can degenerate to $O(n)$ running time if they are out of balance.
- 2-3 trees and B-trees are *balanced* search trees that guarantee $O(\log n)$ performance.
- When data is stored on disk, the most important performance consideration is reducing the number of disk accesses.
- B-trees offer improved performance for on-disk data dictionaries.

Heaps and Priority Queues

Computer Science E-22
Harvard University

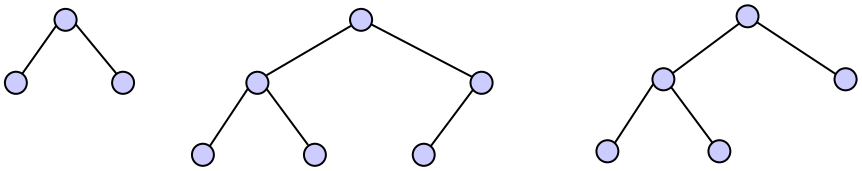
David G. Sullivan, Ph.D.

Priority Queue

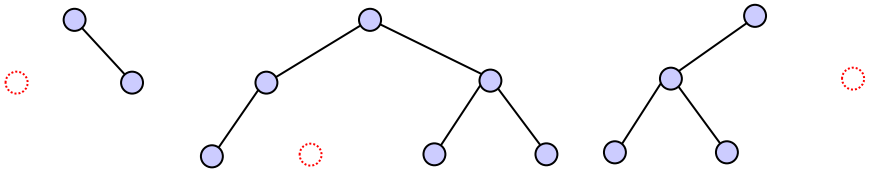
- A *priority queue* (PQ) is a collection in which each item has an associated number known as a *priority*.
  - ("Ann Cudd", 10), ("Robert Brown", 15), ("Dave Sullivan", 5)
  - use a higher priority for items that are "more important"
- Example application: scheduling a shared resource like the CPU
  - give some processes/applications a higher priority, so that they will be scheduled first and/or more often
- Key operations:
  - *insert:* add an item (with a position based on its priority)
  - *remove:* remove the item with the highest priority
- One way to implement a PQ efficiently is using a type of binary tree known as a *heap.*

Complete Binary Trees

- A binary tree of height *h* is *complete* if:
  - levels 0 through *h* - 1 are fully occupied
  - there are no “gaps” to the left of a node in level *h*
- Complete:

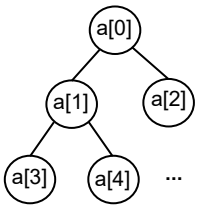


- Not complete (◌ = missing node):

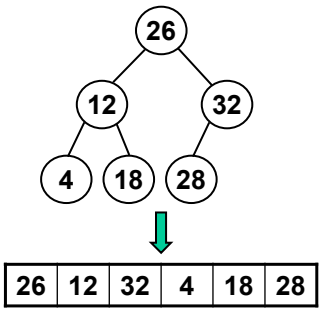


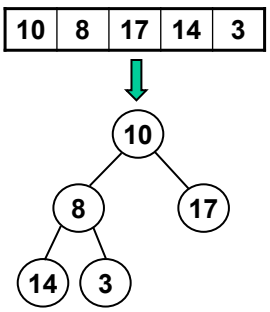
Representing a Complete Binary Tree

- A complete binary tree has a simple array representation.
- The tree's nodes are stored in the array in the order given by a level-order traversal.
  - top to bottom, left to right



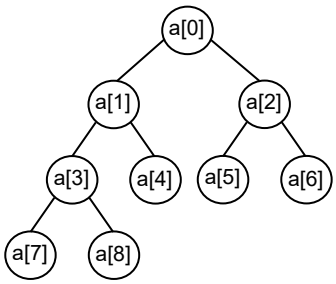
- Examples:





Navigating a Complete Binary Tree in Array Form

- The root node is in `a[0]`
- Given the node in `a[i]`:
  - its left child is in `a[2*i + 1]`
  - its right child is in `a[2*i + 2]`
  - its parent is in `a[(i - 1)/2]` (using integer division)



- Examples:
  - the left child of the node in `a[1]` is in `a[2*1 + 1]` = `a[3]`
  - the left child of the node in `a[2]` is in `a[2*2 + 1]` = `a[5]`
  - the right child of the node in `a[3]` is in `a[2*3 + 2]` = `a[8]`
  - the right child of the node in `a[2]` is in _____
  - the parent of the node in `a[4]` is in `a[(4-1)/2]` = `a[1]`
  - the parent of the node in `a[7]` is in _____

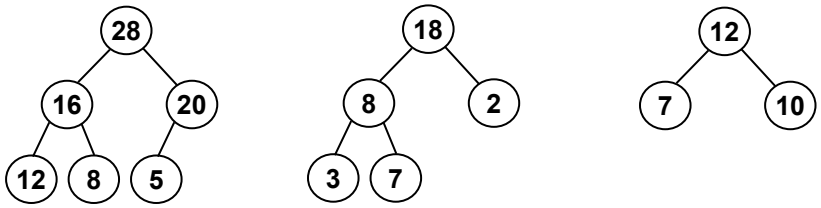
What is the left child of 24?

- Assume that the following array represents a complete tree:

0	1	2	3	4	5	6	7	8
26	**12**	**32**	**24**	**18**	**28**	**47**	**10**	**9**

Heaps

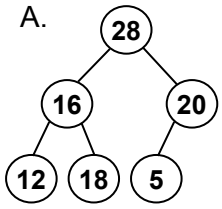
- Heap: a complete binary tree in which each interior node is greater than or equal to its children
  - examples:



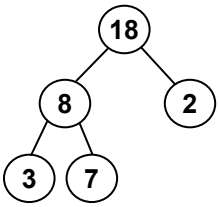
- The largest value is always at the root of the tree.
- The smallest value can be in *any* leaf node - there's no guarantee about which one it will be.
- We're using *max-at-top* heaps.
  - in a *min-at-top* heap, every interior node <= its children

Which of these is a heap?

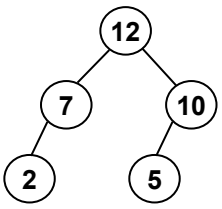
- A.



B.



C.



D. more than one (which ones?)

E. none of them

How to Compare Objects

- We need to be able to compare items in the heap.
- If those items are objects, we can't just do something like this:

```
if (item1 < item2)
```

  Why not?

- Instead, we need to use a method to compare them.

An Interface for Objects That Can Be Compared

- The `Comparable` interface is a built-in generic Java interface:

```
public interface Comparable<T> {
    public int compareTo(T other);
}
```

- It is used when defining a class of objects that can be ordered.
- Examples from the built-in Java classes:

```
public class String implements Comparable<String> {
    ...
    public int compareTo(String other) {
        ...
}
public class Integer implements Comparable<Integer> {
    ...
    public int compareTo(Integer other) {
        ...
}
```

An Interface for Objects That Can Be Compared (cont.)

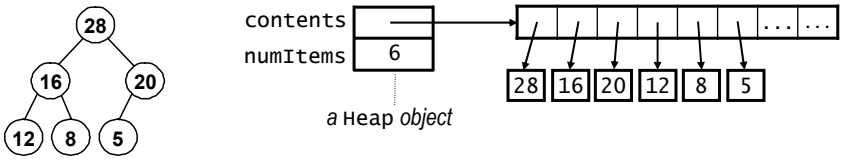
```
public interface Comparable<T> {
    public int compareTo(T other);
}
```

- `item1.compareTo(item2)` should return:
  - a negative integer if `item1` "comes before" `item2`
  - a positive integer if `item1` "comes after" `item2`
  - 0 if `item1` and `item2` are equivalent in the ordering
- These conventions make it easy to construct appropriate method calls:

numeric comparison	comparison using `compareTo`
`item1 < item2`	`item1.compareTo(item2) < 0`
`item1 > item2`	`item1.compareTo(item2) > 0`
`item1 == item2`	`item1.compareTo(item2) == 0`

Heap Implementation

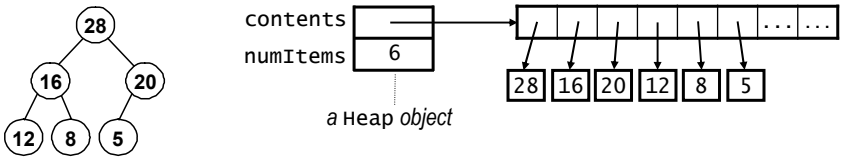
```
public class Heap<T extends Comparable<T>> {
    private T[] contents;
    private int numItems;

    public Heap(int maxSize) {
        contents = (T[])new Comparable[maxSize];
        numItems = 0;
    }
    …
}
```



- `Heap` is another example of a generic collection class.
  - as usual, `T` is the type of the elements
  - `extends Comparable<T>` specifies `T` must implement `Comparable<T>`
  - must use `Comparable` (not `Object`) when creating the array

Heap Implementation (cont.)

```
public class Heap<T extends Comparable<T>> {
    private T[] contents;
    private int numItems;

    …
}
```



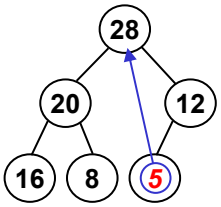
- The picture above is a heap of integers:

  `Heap<Integer> myHeap = new Heap<Integer>(20);`

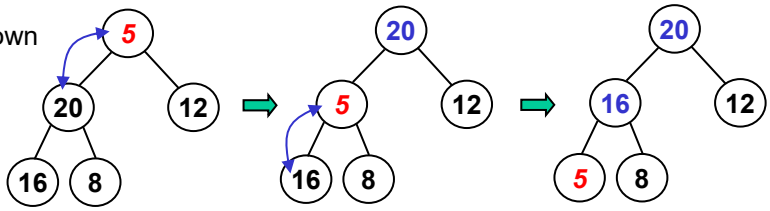
  - works because `Integer` implements `Comparable<Integer>`
  - could also use `String` or `Double`

Removing the Largest Item from a Heap

- Remove and return the item in the root node.
- In addition, need to move the largest remaining item to the root, while maintaining a complete tree with each node >= children
- Algorithm:
  1. make a copy of the largest item
  2. move the last item in the heap to the root
  3. "sift down" the new root item until it is >= its children (or it's a leaf)
  4. return the largest item



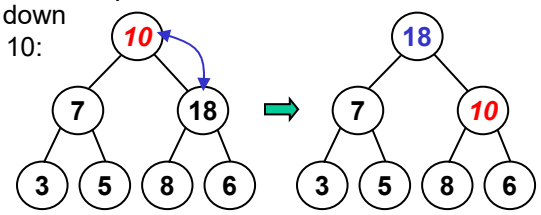
sift down the 5:



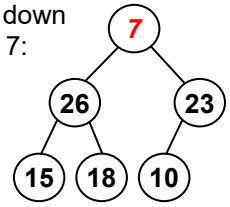
Sifting Down an Item

- To sift down item *x* (i.e., the item whose key is *x*):
  1. compare *x* with the larger of the item's children, *y*
  2. if *x* < *y*, swap *x* and *y* and repeat
- Other examples:

sift down the 10:



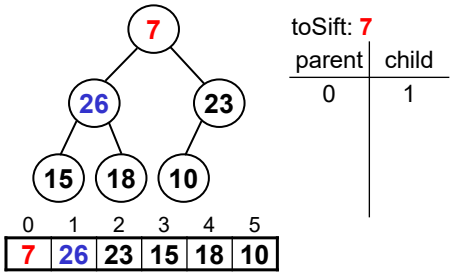
sift down the 7:



`siftDown()` Method

```
private void siftDown(int i) {    // assume i = 0
    T toSift = contents[i];

    int parent = i;
    int child = 2 * parent + 1;
    while (child < numItems) {
        // If the right child is bigger, set child to be its index.
        if (child < numItems - 1  &&
          contents[child].compareTo(contents[child + 1]) < 0) {
            child = child + 1;
        }
        if (toSift.compareTo(contents[child]) >= 0) {
            break;  // we're done
        }
        // Move child up and move down one level in the tree.
        contents[parent] = contents[child];
        parent = child;
        child = 2 * parent + 1;
    }

    contents[parent] = toSift;
}
```

- We don't actually swap items. We put the sifted item in place at the end.

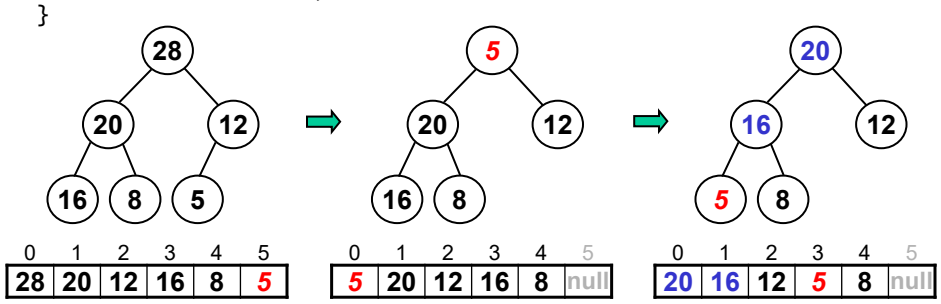


toSift: 7

parent	child
0	1

remove() Method

```
public T remove() {
    // check for empty heap goes here
    T toRemove = contents[0];

    contents[0] = contents[numItems - 1];
    contents[numItems - 1] = null;
    numItems--;
    siftDown(0);

    return toRemove;
}
```



numItems: **6**
toRemove: **28**

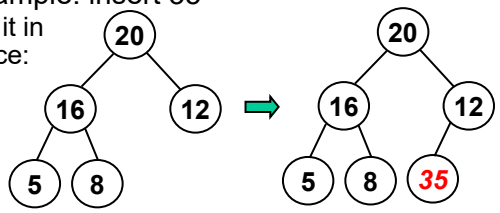
numItems: **5**
toRemove: **28**

numItems: **5**
toRemove: **28**

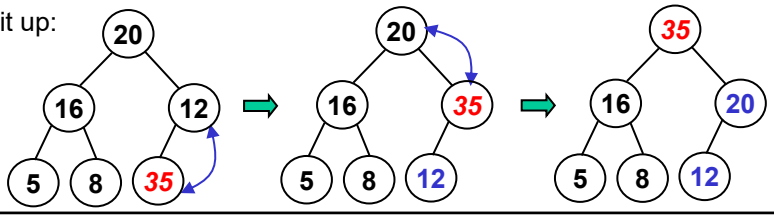
Inserting an Item in a Heap

- Algorithm:
  1. put the item in the next available slot (grow array if needed)
  2. "sift up" the new item
     until it is <= its parent (or it becomes the root item)
- Example: insert 35

put it in place:

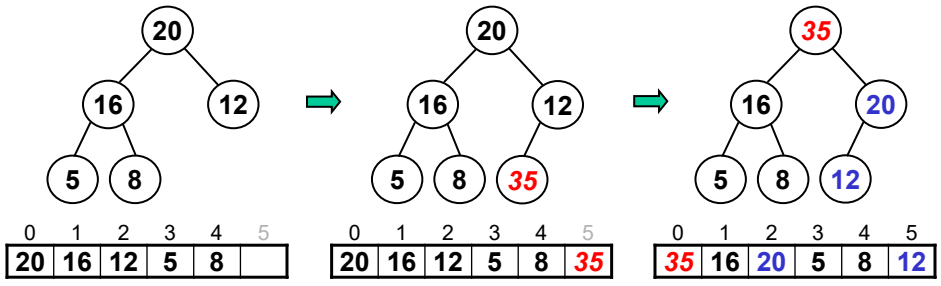


sift it up:



`insert()` Method

```
public void insert(T item) {
    if (numItems == contents.length) {
        // code to grow the array goes here…
    }
    contents[numItems] = item;
    siftUp(numItems);
    numItems++;
}
```

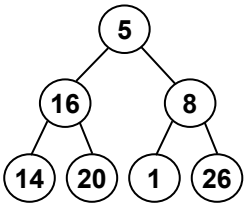


numItems: **5**
item: **35**

numItems: **5**
item: **35**

numItems: **6**

Time Complexity of a Heap



- A heap containing n items has a height <= $\log_2 n$. Why?
- Thus, removal and insertion are both $O(\log n)$.
  - remove: go down at most $\log_2 n$ levels when sifting down; do a constant number of operations per level
  - insert: go up at most $\log_2 n$ levels when sifting up; do a constant number of operations per level
- This means we can use a heap for a $O(\log n)$-time priority queue.

Using a Heap for a Priority Queue

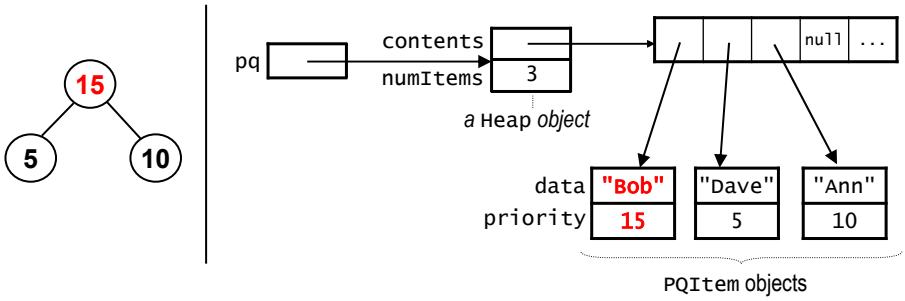
- Recall: a *priority queue* (PQ) is a collection in which each item has an associated number known as a *priority*.
  - ("Ann Cudd", 10), ("Robert Brown", 15), ("Dave Sullivan", 5)
  - use a higher priority for items that are "more important"
- To implement a PQ using a heap:
  - order the items in the heap according to their priorities
    - every item in the heap will have a priority >= its children
    - the highest priority item will be in the root node
  - get the highest priority item by calling `heap.remove()`!
- For this to work, we need a "wrapper" class for items that we put in the priority queue.
  - will group together an item with its priority
  - with a `compareTo()` method that compares priorities!

A Class for Items in a Priority Queue

```
public class PQItem implements Comparable<PQItem> {
    // group an arbitrary object with a priority
    private Object data;
    private int priority;
    ...

    public int compareTo(PQItem other) {
        // error-checking goes here…
        return (priority - other.priority);
    }
}
```

- Example: `PQItem item = new PQItem("Dave Sullivan", 5);`
- Its `compareTo()` compares `PQItem`s based on their priorities.
- `item1.compareTo(item2)` returns:
  - a negative integer if `item1` has a lower priority than `item2`
  - a positive integer if `item1` has a higher priority than `item2`
  - 0 if they have the same priority

Using a Heap for a Priority Queue



- Sample client code:

```
Heap<PQItem> pq = new Heap<PQItem>(50);
pq.insert(new PQItem("Dave", 5));
pq.insert(new PQItem("Ann", 10));
pq.insert(new PQItem("Bob", 15));

PQItem mostImportant = pq.remove(); // will get Bob!
```

Using a Heap to Sort an Array

- Recall selection sort: it repeatedly finds the smallest remaining element and swaps it into place:

0	1	2	3	4	5	6
5	**16**	**8**	**14**	**20**	**1**	**26**
1	**16**	**8**	**14**	**20**	**5**	**26**
1	**5**	**8**	**14**	**20**	**16**	**26**

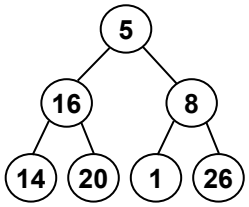
...

- It isn't efficient, because it performs a linear scan to find the smallest remaining element (*O*(n) steps per scan).
- Heapsort is a sorting algorithm that repeatedly finds the *largest* remaining element and puts it in place.
- It *is* efficient, because it turns the array into a heap.
  - it can find/remove the largest remaining in *O*(log n) steps!

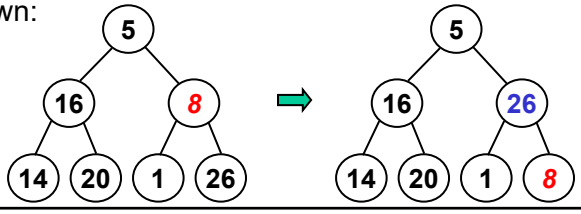
Converting an Arbitrary Array to a Heap

- To convert an array (call it `contents`) with n items to a heap:
  1. start with the parent of the last element:
     `contents[i]`, where `i = ((n – 1) – 1)/2 = (n – 2)/2`
  2. sift down `contents[i]` and all elements to its left
- Example:

0	1	2	3	4	5	6
5	16	8	14	20	1	26

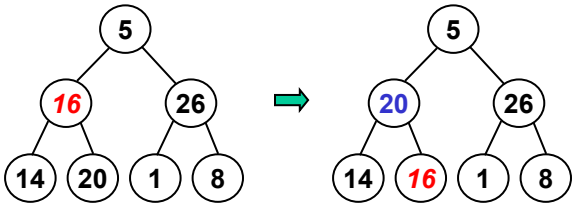


- Last element's parent = `contents[(7 – 2)/2]` = `contents[2]`. Sift it down:

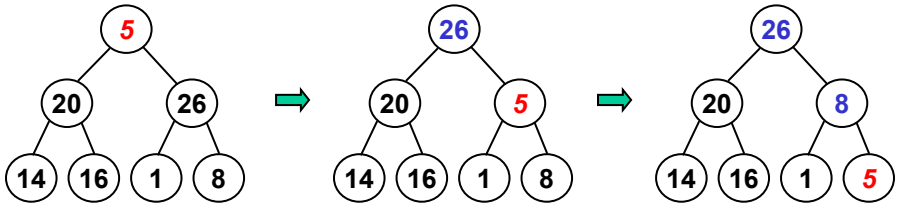


Converting an Array to a Heap (cont.)

- Next, sift down `contents[1]`:



- Finally, sift down `contents[0]`:



Heapsort

- Pseudocode:

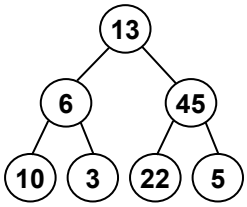
```
heapSort(arr) {
    // Turn the array into a max-at-top heap.
    heap = new Heap(arr);

    endUnsorted = arr.length - 1;
    while (endUnsorted > 0) {
       // Get the largest remaining element and put it
       // at the end of the unsorted portion of the array.
       largestRemaining = heap.remove();
       arr[endUnsorted] = largestRemaining;

       endUnsorted--;
    }
}
```

Heapsort Example

- Sort the following array:

0	1	2	3	4	5	6
13	6	45	10	3	22	5

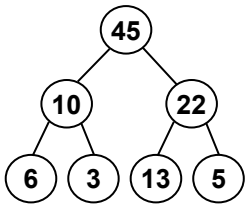
- Here's the corresponding complete tree:

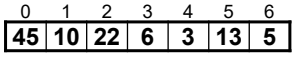


- Begin by converting it to a heap:

Heapsort Example (cont.)

- Here's the heap in both tree and array forms:





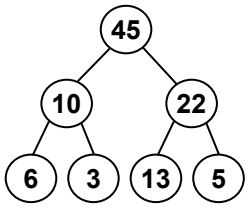
endUnsorted: **6**

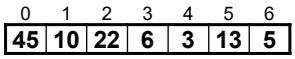
- We begin looping:

```
while (endUnsorted > 0) {
    // Get the largest remaining element and put it
    // at the end of the unsorted portion of the array.
    largestRemaining = heap.remove();
    arr[endUnsorted] = largestRemaining;

    endUnsorted--;
}
```

Heapsort Example (cont.)

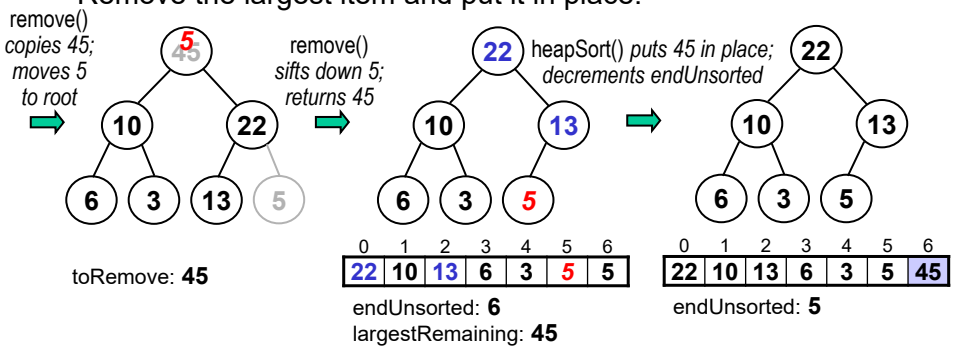
- Here's the heap in both tree and array forms:

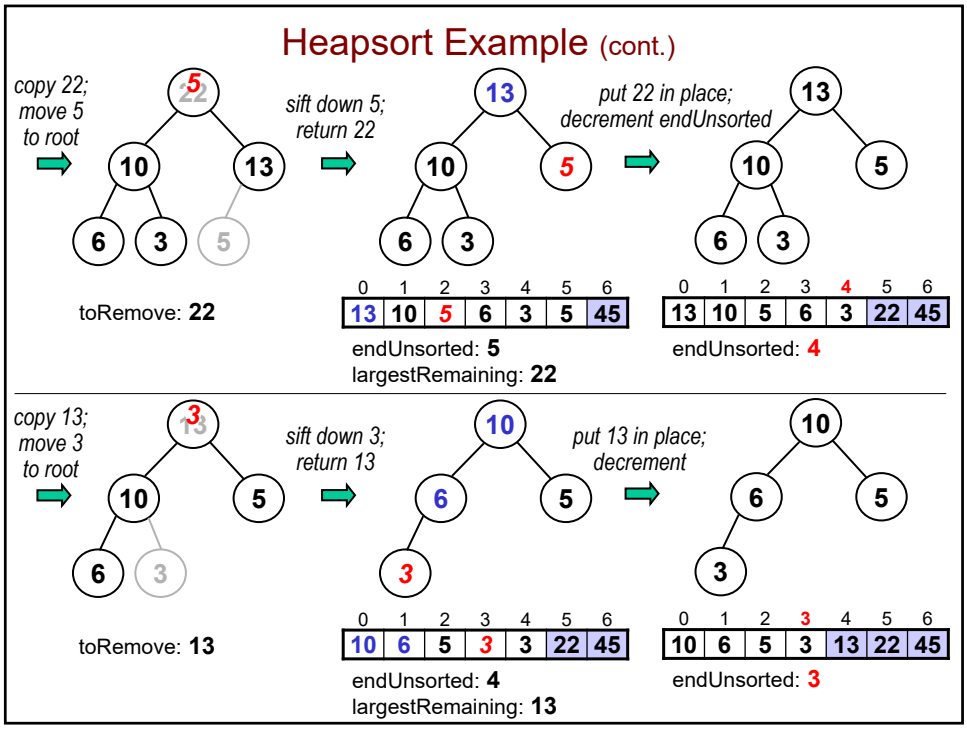


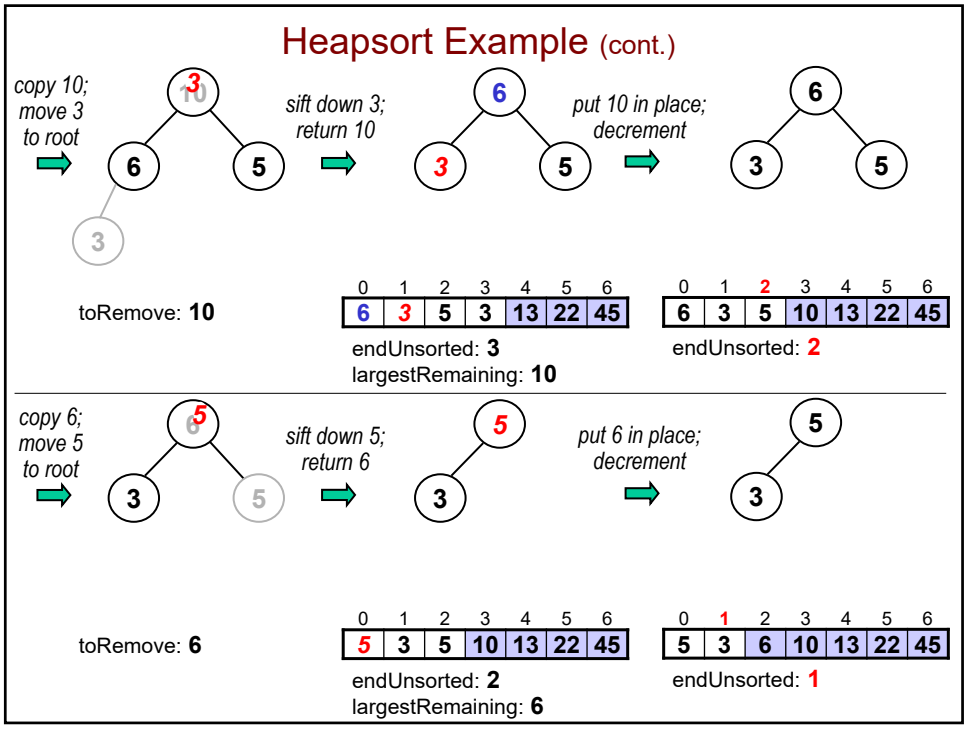


endUnsorted: **6**

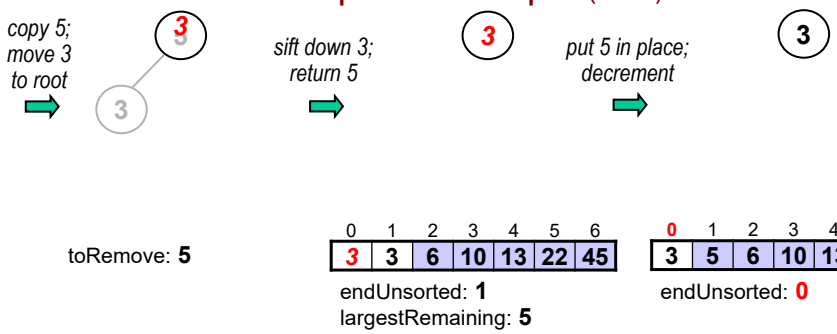
- Remove the largest item and put it in place:




Heapsort Example (cont.)
copy 22; move 5 to root
sift down 5; return 22
put 22 in place; decrement endUnsorted
toRemove: 22
endUnsorted: 5
largestRemaining: 22
endUnsorted: 4
copy 13; move 3 to root
sift down 3; return 13
put 13 in place; decrement
toRemove: 13
endUnsorted: 4
largestRemaining: 13
endUnsorted: 3


Heapsort Example (cont.)
copy 10; move 3 to root
sift down 3; return 10
put 10 in place; decrement
toRemove: 10
endUnsorted: 3
largestRemaining: 10
endUnsorted: 2
copy 6; move 5 to root
sift down 5; return 6
put 6 in place; decrement
toRemove: 6
endUnsorted: 2
largestRemaining: 6
endUnsorted: 1

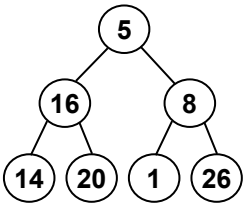
Heapsort Example (cont.)



- And now we terminate the loop:

```
while (endUnsorted > 0) {
    // Get the largest remaining element and put it
    // at the end of the unsorted portion of the array.
    largestRemaining = heap.remove();
    arr[endUnsorted] = largestRemaining;

    endUnsorted--;
}
```

Efficiency of Heapsort



- Time complexity of going from a heap to a sorted array?

- It can be shown that turning an array into a heap takes O(n) steps.
  - even better than O(n log n)!
  - n/2 calls to siftDown(), most of which involve small subheaps
- Thus, total time complexity = *?*

How Does Heapsort Compare?

algorithm	best case	avg case	worst case	extra memory
selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$
Shell sort	$O(n \log n)$	$O(n^{1.5})$	$O(n^{1.5})$	$O(1)$
bubble sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
quicksort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$O(\log n)$ worst: $O(n)$
mergesort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$
heapsort	$\mathbf{O(n \log n)}$	$\mathbf{O(n \log n)}$	$\mathbf{O(n \log n)}$	$\mathbf{O(1)}$

- Heapsort matches mergesort for the best worst-case time complexity, but it has better space complexity.
- Insertion sort is still best for arrays that are almost sorted.
- Quicksort is still typically fastest in the average case.

Hash Tables

Computer Science E-22
Harvard University

David G. Sullivan, Ph.D.

Data Dictionary Revisited

- We've considered several data structures that allow us to store and search for data items using their key fields:

data structure	***searching for an item***	***inserting an item***
a list implemented using an array	$O(\log n)$ using binary search	$O(n)$
a list implemented using a linked list	$O(n)$ using linear search	$O(n)$
binary search tree		
balanced search trees (2-3 tree, B-tree, others)		

- We'll now look at hash tables, which can do better than $O(\log n)$.

Ideal Case: Searching = Indexing

- We would achieve optimal efficiency if we could treat the key as an index into an array.
- Example: storing data about members of a sports team
  - key = jersey number (some value from 0-99).
  - class for an individual player's record:

```
public class Player {
    private int jerseyNum;
    private String firstName;
    …
}
```

  - store the player records in an array:

```
Player[] teamRecords = new Player[100];
```

- In such cases, search and insertion are *O*(1):

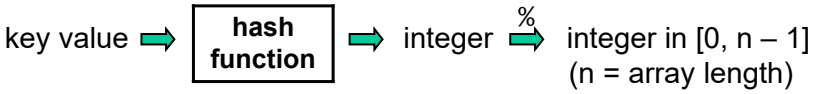
```
public Player search(int jerseyNum) {
    return teamRecords[jerseyNum];
}
```

Hashing: Turning Keys into Array Indices

- In most real-world problems, indexing is not as simple as the sports-team example. Why?
  -
  -
  -
- To handle these problems, we perform *hashing*:
  - use a *hash function* to convert the keys into array indices
    `"Sullivan"` → `18`
  - use techniques to handle cases in which multiple keys are assigned the same hash value
- The resulting data structure is known as a *hash table*.

Hash Functions

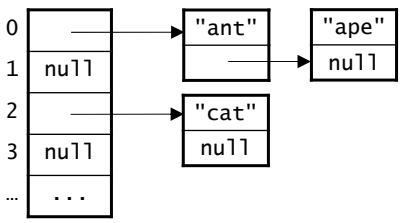
- A hash function defines a mapping from keys to integers.
- We then use the modulus operator to get a valid array index.



- Here's a very simple hash function for keys of lower-case letters:
  h(key) = ASCII value of first char – ASCII value of 'a'
  - examples:
    h("ant") = ASCII for 'a' – ASCII for 'a' = 0
    h("cat") = ASCII for 'c' – ASCII for 'a' = 2
- h(key) is known as the key's *hash code.*
- A *collision* occurs when items with different keys are assigned the same hash code.

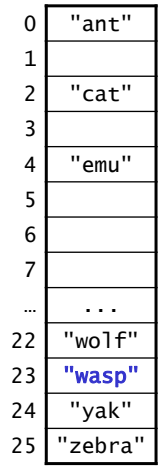
Dealing with Collisions I: Separate Chaining

- Each position in the hash table serves as a *bucket* that can store multiple data items.
- Two options:
  1. each bucket is itself an array
     - need to preallocate, and a bucket may become full
  2. each bucket is a linked list
     - items with the same hash code are "chained" together
     - each "chain" can grow as needed



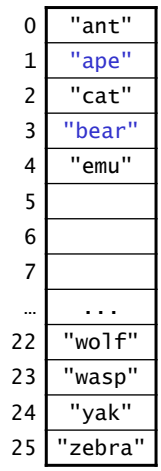
Dealing with Collisions II: Open Addressing

- When the position assigned by the hash function is occupied, find another open position.
- Example: "wasp" has a hash code of 22, but it ends up in position 23 because position 22 is occupied.
- We'll consider three ways of finding an open position – a process known as *probing*.
- We also perform probing when searching.
  - example: search for "wasp"
    - look in position 22
    - then look in position 23
  - need to figure out when to safely stop searching (more on this soon!)



Linear Probing

- Probe sequence: h(key), h(key) + 1, h(key) + 2, …, wrapping around as necessary.
- Examples:
  - "ape" (h = 0) would be placed in position 1, because position 0 is already full.
  - "bear" (h = 1): try 1, 1 + 1, 1 + 2 – open!
  - where would "zebu" end up?
- Advantage: if there is an open cell, linear probing will eventually find it.
- Disadvantage: get "clusters" of occupied cells that lead to longer subsequent probes.
  - probe length = the number of positions considered during a probe



Quadratic Probing

- Probe sequence: h(key), h(key) + 1^2, h(key) + 2^2, h(key) + 3^2, …, wrapping around as necessary.
- Examples:
  - "ape" (h = 0): try 0, 0 + 1 – open!
  - "bear" (h = 1): try 1, 1 + 1, 1 + 4 – open!
  - "zebu"?
- Advantage: smaller clusters of occupied cells
- Disadvantage: may fail to find an existing open position. For example:

table size = 10
x = occupied

trying to insert a key with h(key) = 0

offsets of the probe sequence in italics

0	x	
1	x	*1 81*
2		
3		
4	x	*4 64*
5	x	*25*
6	x	*16 36*
7		
8		
9	x	*9 49*

0	"ant"
1	"ape"
2	"cat"
3	
4	"emu"
5	"bear"
6	
7	
…	...
22	"wolf"
23	"wasp"
24	"yak"
25	"zebra"

Double Hashing

- Use two hash functions:
  - h1 computes the hash code
  - h2 computes the increment for probing
  - probe sequence: h1, h1 + h2, h1 + 2*h2, …
- Examples:
  - h1 = our previous h
  - h2 = number of characters in the string
  - "ape" (h1 = 0, h2 = 3): try 0, 0 + 3 – open!
  - "bear" (h1 = 1, h2 = 4): try 1 – open!
  - "zebu"?
- Combines good features of linear and quadratic:
  - reduces clustering
  - will find an open position if there is one, provided the table size is a prime number

0	"ant"
1	"bear"
2	"cat"
3	"ape"
4	"emu"
5	
6	
7	
…	...
22	"wolf"
23	"wasp"
24	"yak"
25	"zebra"

Removing Items Under Open Addressing

- Problematic example (using linear probing):
  - insert "ape" (h = 0): try 0, 0 + 1 – open!
  - insert "bear" (h = 1): try 1, 1 + 1, 1 + 2 – open!
  - remove "ape"
  - search for "ape": try 0, 0 + 1 – conclude not in table
  - search for "bear": **try 1 – conclude not in table, but "bear" is further down in the table!**

0	"ant"
1	
2	"cat"
3	"bear"
4	"emu"
5	
…	...
22	"wolf"
23	"wasp"
24	"yak"
25	"zebra"

- To fix this problem, distinguish between:
  - *removed positions* that previously held an item
  - *empty positions* that have never held an item
- During probing, we *don't* stop if we see a removed position.
  ex: search for "bear": try 1 (removed), 1 + 1, **1 + 2 – found!**
- We can insert items in either empty or removed positions.

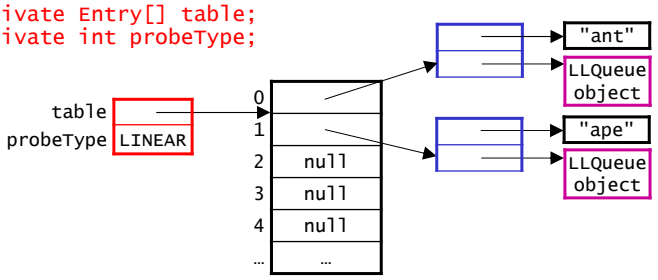
An Interface For Hash Tables

```
public interface HashTable {
    boolean insert(Object key, Object value);
    Queue<Object> search(Object key);
    Queue<Object> remove(Object key);
}
```

- `insert()` takes a key-value pair and returns:
  - `true` if the key-value pair can be added
  - `false` if it cannot be added (referred to as *overflow*)
- `search()` and `remove()` both take a key, and return a queue containing all of the values associated with that key.
  - example: an index for a book
    - key = word
    - values = the pages on which that word appears
  - return `null` if the key is not found

An Implementation Using Open Addressing

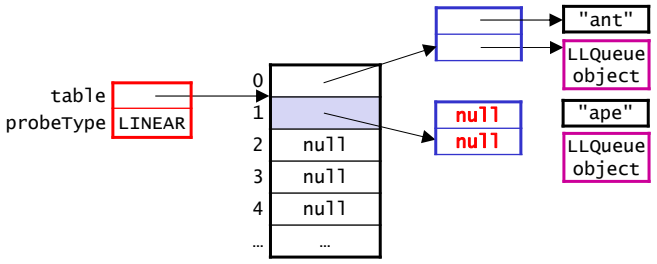
```
public class OpenHashTable implements HashTable {
    private class Entry {
        private Object key;
        private LLQueue<Object> values;
        …
    }
    …
    private Entry[] table;
    private int probeType;
}
```



- We use a private inner class for the entries in the hash table.
- We use an `LLQueue` for the values associated with a given key.

Empty vs. Removed

- When we remove a key and its values, we:
  - leave the `Entry` object in the table
  - set the `Entry` object's `key` and `values` fields to `null`
  - example: after `remove("ape")`:



- Note the difference:
  - a truly empty position has a value of `null` in the table (example: positions 2, 3 and 4 above)
  - a removed position refers to an `Entry` object whose `key` and `values` fields are `null` (example: position 1 above)

Probing Using Double Hashing

```
private int probe(Object key) {
    int i = h1(key);    // first hash function
    int h2 = h2(key);   // second hash function

    // keep probing until we get an empty position or match
    while (table[i] != null && !key.equals(table[i].key)) {
        i = (i + h2) % table.length;
    }

    return i;
}
```

- It is essential that we:
  - check for `table[i] != null` first. why?
  - call the `equals` method on `key`, not `table[i].key`. why?

Avoiding an Infinite Loop

- The while loop in our probe method could lead to an infinite loop.

```
while (table[i] != null && !key.equals(table[i].key)) {
    i = (i + h2) % table.length;
}
```

- When would this happen?
- We can stop probing after checking n positions (n = table size), because the probe sequence will just repeat after that point.
  - for quadratic probing:
    $(h1 + n^2) \% n = h1 \% n$
    $(h1 + (n+1)^2) \% n = (h1 + n^2 + 2n + 1) \% n = (h1 + 1) \% n$
  - for double hashing:
    $(h1 + n*h2) \% n = h1 \% n$
    $(h1 + (n+1)*h2) \% n = (h1 + n*h2 + h2) \% n = (h1 + h2) \% n$

Avoiding an Infinite Loop (cont.)

```
private int probe(Object key) {
    int i = h1(key);    // first hash function
    int h2 = h2(key);   // second hash function
    int numChecked = 1;

    // keep probing until we get an empty position or a match
    while (table[i] != null && !key.equals(table[i].key)) {
        if (numChecked == table.length) {
            return -1;
        }
        i = (i + h2) % table.length;
        numChecked++;
    }

    return i;
}
```

Search and Removal

```
  public LLQueue<Object> search(Object key) {
      // throw an exception if key == null
      int i = probe(key);
      if (i == -1 || table[i] == null) {
          return null;
      } else {
          return table[i].values;
      }
  }

  public LLQueue<Object> remove(Object key) {
      // throw an exception if key == null
      int i = probe(key);
      if (i == -1 || table[i] == null) {
          return null;
      }

      LLQueue<Object> removedVals = table[i].values;
      table[i].key = null;
      table[i].values = null;
      return removedVals;
  }
```

Insertion

- We begin by probing for the key.
- Several cases:
  1. the key is already in the table (we're inserting a duplicate)
     → add the value to the values in the key's `Entry`
  2. the key is not in the table: three subcases:
     a. encountered 1 or more removed positions while probing
        → put the (key, value) pair in the *first* removed position seen during probing. why?

     b. no removed position; reached an empty position
        → put the (key, value) pair in the empty position

     c. no removed position or empty position
        → overflow; return `false`

Tracing Through Some Examples

- Start with the hash table at right with:
  - double hashing
  - our earlier hash functions h1 and h2
- Perform the following operations:
  - insert "bear" (h1 = 1, h2 = 4):
  - insert "bison" (h1 = 1, h2 = 5):
  - insert "cow" (h1 = 2, h2 = 3):
  - delete "emu" (h1 = 4, h2 = 3):
  - search "eel" (h1 = 4, h2 = 3):
  - insert "bee" (h1 = ___, h2 = ____):

0	"ant"
1	
2	"cat"
3	
4	"emu"
5	"fox"
6	
7	
8	
9	
10	

Dealing with Overflow

- Overflow = can't find a position for an item
- When does it occur?
  - linear probing:
  - quadratic probing:
    -
    -
  - double hashing:
    - if the table size is a prime number: same as linear
    - if the table size is not a prime number: same as quadratic
- To avoid overflow (and reduce search times), grow the hash table when the % of occupied positions gets too big.
  - problem: we need to rehash **all** of the existing items. why?

Implementing the Hash Function

- Characteristics of a good hash function:
  1) efficient to compute
  2) uses the entire key
     - changing any char/digit/etc. should change the hash code
  3) distributes the keys more or less uniformly across the table
  4) must be a function!
     - a key must always get the same hash code
- In Java, every object has a `hashCode()` method.
  - the version inherited from `Object` returns a value based on an object's memory location
  - classes can override this version with their own

Hash Functions for Strings: version 1

- h_a = the sum of the characters' ASCII values
  - example: h_a("eat") = 101 + 97 + 116 = 314
- All permutations of a given set of characters get the same code.
  - example: h_a("tea") = h_a("eat")
  - could be useful in a Scrabble game
    - allow you to look up all words that can be formed from a given set of characters
- The range of possible hash codes is very limited.
  - example: hashing keys composed of 1-5 lower-case char's (padded with spaces)
  - 26*27*27*27*27 = over 13 million possible keys
  - smallest code = h_a("a ") = 97 + 4*32 = 225
    largest code = h_a("zzzzz") = 5*122 = 610
    } 610 – 225 = 385 codes

Hash Functions for Strings: version 2

- Compute a *weighted* sum of the ASCII values:

$$h_b = a_0 b^{n-1} + a_1 b^{n-2} + \ldots + a_{n-2} b + a_{n-1}$$

where a_i = ASCII value of the ith character
b = a constant
n = the number of characters

- Multiplying by powers of b allows the *positions* of the characters to affect the hash code.
  - different permutations get different codes
- We may get arithmetic overflow, and thus the code may be negative. We adjust it when this happens.
- Java uses this hash function with b = 31 in the `hashCode()` method of the `String` class.

Hash Table Efficiency

- In the best case, search and insertion are *O*(1).
- In the worst case, search and insertion are linear.
  - open addressing: *O*(m), where m = the size of the hash table
  - separate chaining: *O*(n), where n = the number of keys
- With good choices of hash function and table size, complexity is generally better than *O*(log n) and approaches *O*(1).
- *load factor* = # keys in table / size of the table.
  To prevent performance degradation:
  - open addressing: try to keep the load factor < 1/2
  - separate chaining: try to keep the load factor < 1
- Time-space tradeoff: bigger tables have better performance, but they use up more memory.

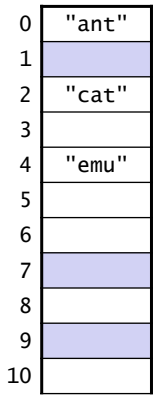
Hash Table Limitations

- It can be hard to come up with a good hash function for a particular data set.
- The items are not ordered by key. As a result, we can't easily:
  - print the contents in sorted order
  - perform a range search (find all values between v1 and v2)
  - perform a rank search – get the kth largest item

  We *can* do all of these things with a search tree.

Extra Practice

- Start with the hash table at right with:
  - double hashing
  - h1(key) = ASCII of first letter – ASCII of 'a'
  - h2(key) = key.length()
  - shaded cells are removed cells
- **What is the probe sequence for "baboon"?**
  (the sequence of positions seen during probing)



A. 1, 2, 5
B. 1, 6
C. 1, 7, 2
D. 1, 7, 3
E. 1, 7, 2, 8

Extra Practice

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  - double hashing
  - h1(key) = ASCII of first letter – ASCII of 'a'
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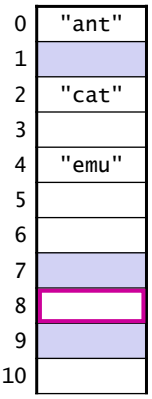
- **What is the probe sequence for "baboon"?**
  (h1 = 1, h2 = 6) try: 1 % 11 = **1**
  (1 + 6) % 11 = **7**
  (1 + 2*6) % 11 = **2**
  (1 + 3*6) % 11 = **8**
  empty cell, so stop probing

A. 1, 2, 5
B. 1, 6
C. 1, 7, 2
D. 1, 7, 3
E. **1, 7, 2, 8**



Extra Practice

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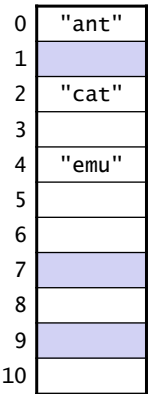
  (h1 = 1, h2 = 6) try: 1 % 11 = 1
  (1 + 6) % 11 = 7
  (1 + 2*6) % 11 = 2
  (1 + 3*6) % 11 = 8
- **If we insert "baboon", in what position will it go?**

A. 1 B. 7 C. 2 D. 8



Extra Practice

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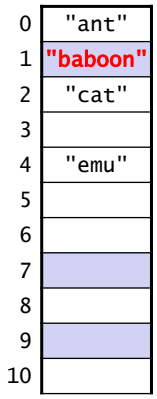
  - double hashing
  - h1(key) = ASCII of first letter – ASCII of 'a'
  - h2(key) = key.length()
  - shaded cells are removed cells



- What is the probe sequence for "baboon"?
  (h1 = 1, h2 = 6) try: 1 % 11 = 1
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  (1 + 2*6) % 11 = 2
  (1 + 3*6) % 11 = 8

- **If we insert "baboon", in what position will it go?**

  A. **1** B. 7 C. 2 D. 8

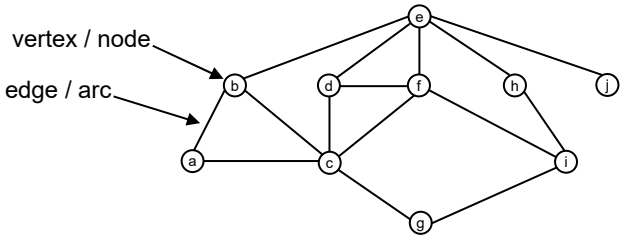
  ↖ the first *removed* position seen while probing

Graphs

Computer Science E-22
Harvard Extension School

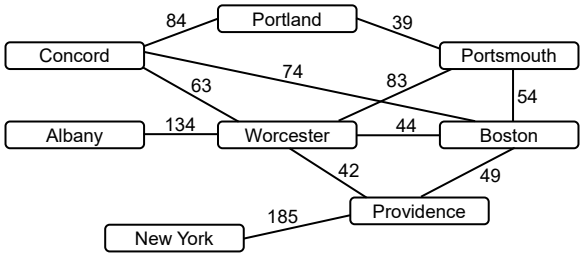
David G. Sullivan, Ph.D.

What is a Graph?



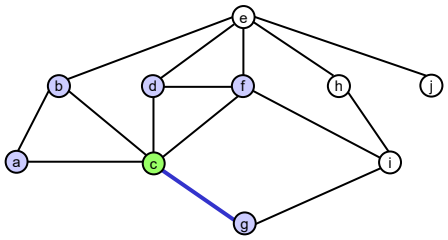
- A graph consists of:
  - a set of *vertices* (also known as *nodes*)
  - a set of *edges* (also known as *arcs*), each of which connects a pair of vertices

Example: A Highway Graph



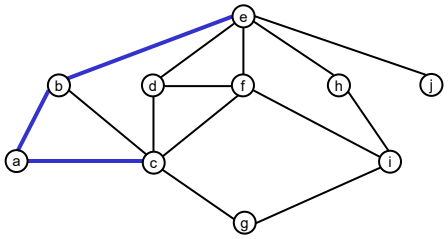
- Vertices represent cities.
- Edges represent highways.
- This is a *weighted* graph, with a *cost* associated with each edge.
  - in this example, the costs denote mileage
- We'll use graph algorithms to answer questions like "What is the shortest route from Portland to Providence?"

Relationships Among Vertices

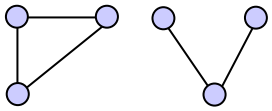


- Two vertices are *adjacent* if they are connected by a single edge.
  - *ex:* c and g are adjacent, but c and i are not
- The collection of vertices that are adjacent to a vertex v are referred to as v's *neighbors.*
  - *ex:* c's neighbors are a, b, d, f, and g

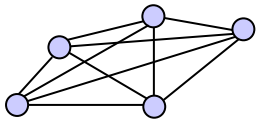
Paths in a Graph



- A *path* is a sequence of edges that connects two vertices.
- A graph is *connected* if there is a path between any two vertices.
  - *ex:* the six vertices at right are part of a graph that is *not* connected

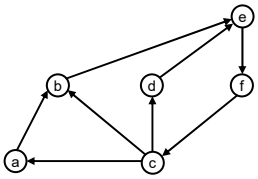


- A graph is *complete* if there is an edge between every pair of vertices.
  - *ex:* the graph at right *is* complete



Directed Graphs

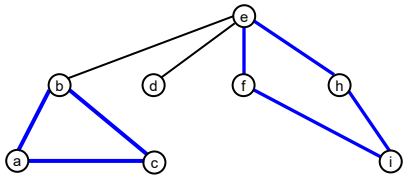
- A *directed* graph has a direction associated with each edge, which is depicted using an arrow:



- Edges in a directed graph are often represented as ordered pairs of the form (start vertex, end vertex).
  - *ex:* (a, b) is an edge in the graph above, but (b, a) is not.
- In a path in a directed graph, the end vertex of edge i must be the same as the start vertex of edge i + 1.
  - *ex:* { (a, **b**), (**b**, **e**), (**e**, f) } is a valid path.
    { (a, **b**), (**c**, **b**), (**c**, a) } is not.

Cycles in a Graph

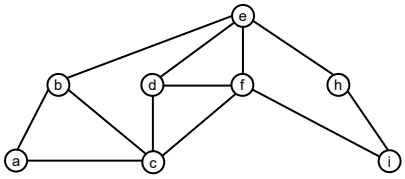
- A *cycle* is a path that:
  - leaves a given vertex using one edge
  - returns to that same vertex using a different edge
- Examples: the highlighted paths below



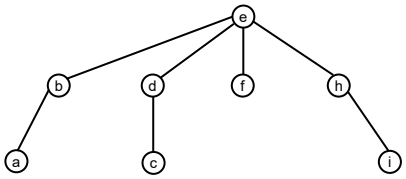
- An *acyclic* graph has no cycles.

Trees vs. Graphs

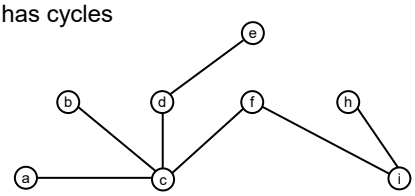
- A tree is a special type of graph.
  - connected, undirected, and acyclic
  - we usually single out one of the vertices to be the root, but graph theory does *not* require this



a graph that is *not* a tree, because it has cycles



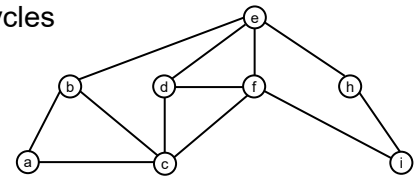
a tree using the same nodes



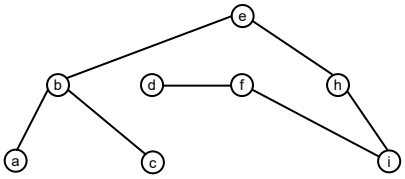
another tree using the same nodes

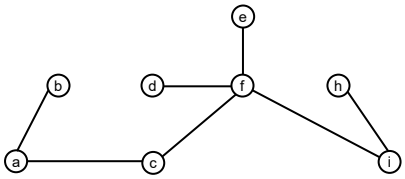
Spanning Trees

- A spanning tree is a subset of a connected graph that contains:
  - all of the vertices
  - a subset of the edges that form a tree
- Recall this graph with cycles from the previous slide:



- The trees on that slide were spanning trees for this graph. Here are two others:

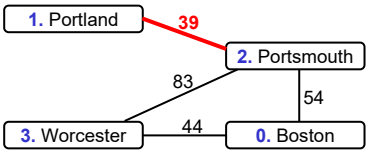




Representing a Graph: Option 1

- Use an *adjacency matrix* – a two-dimensional array in which element [r][c] = the cost of going from vertex r to vertex c
- Example:

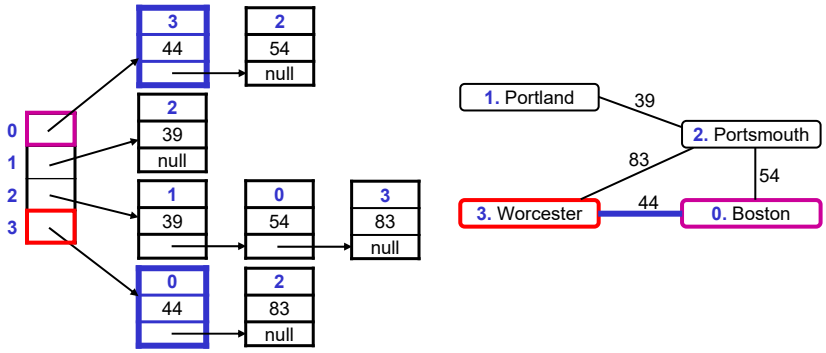
	0	1	2	3
0			54	44
1			39	
2	54	39		83
3	44		83	



- Use a special value to indicate there's no edge from r to c
  - shown as a shaded cell above
  - can't use 0, because an edge may have an actual cost of 0
- This representation:
  - wastes memory if a graph is *sparse* (few edges per vertex)
  - is memory-efficient if a graph is *dense* (many edges per vertex)

Representing a Graph: Option 2

- Use one *adjacency list* for each vertex.
  - a linked list with info on the edges coming from that vertex



- This representation uses less memory if a graph is sparse.
- It uses more memory if a graph is dense.
  - because of the references linking the nodes

Graph Class

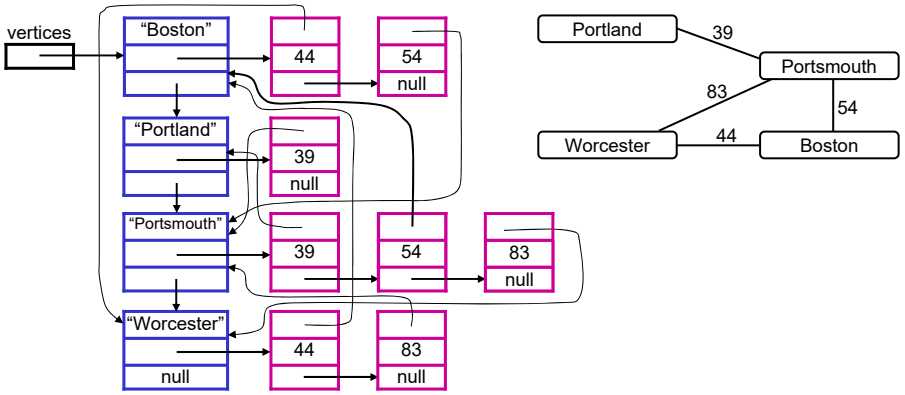
```
public class Graph {
    private class Vertex {
        private String id;
        private Edge edges;              // adjacency list
        private Vertex next;
        private boolean encountered;
        private boolean done;
        private Vertex parent;
        private double cost;
        …
    }

    private class Edge {
        private Vertex start;
        private Vertex end;
        private double cost;
        private Edge next;
        …
    }

    private Vertex vertices;
    …
}
```

The highlighted fields are shown in the diagram on the previous page.

Our Graph Representation



- Each `Vertex` object (shown in blue) stores info. about a vertex.
  - including an adjacency list of `Edge` objects (the purple ones)
- A `Graph` object has a single field called `vertices`
  - a reference to a linked list of `Vertex` objects
  - a linked list of linked lists!

Traversing a Graph

- Traversing a graph involves starting at some vertex and visiting all vertices that can be reached from that vertex.
  - visiting a vertex = processing its data in some way
  - if the graph is connected, all of its vertices will be visited
- We will consider two types of traversals:
  - **depth-first**: proceed as far as possible along a given path before backing up
  - **breadth-first**: visit a vertex
    visit all of its neighbors
    visit all unvisited vertices 2 edges away
    visit all unvisited vertices 3 edges away, etc.

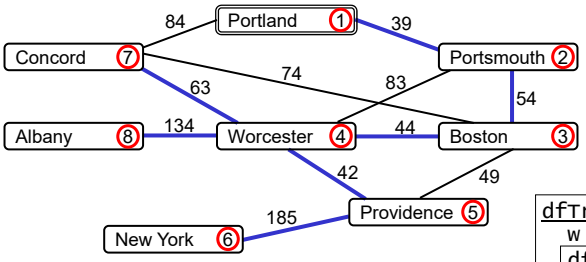
Depth-First Traversal

- Visit a vertex, then make recursive calls on all of its yet-to-be-visited neighbors:

```
private static void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id);  // visit v
    v.done = true;
    v.parent = parent;      // record where we came from

    // walk down v's adjacency list
    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;   // consider each neighbor w
        if (!w.done) {      // if w has not been visited
            dfTrav(w, v);
        }
        e = e.next;
    }
}
```

Example: Depth-First Traversal from Portland

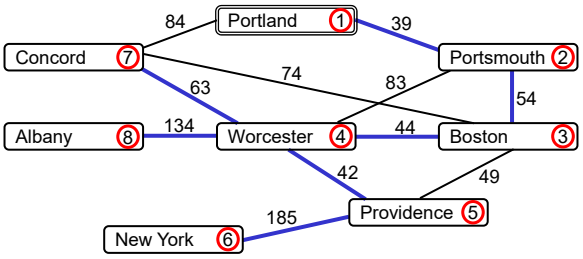


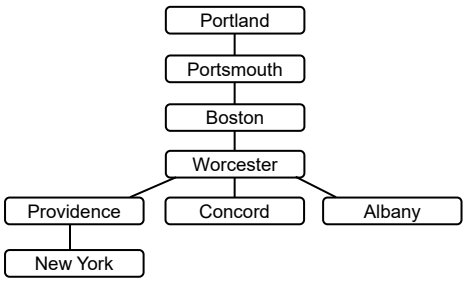
For the examples, we'll assume that the edges in each vertex's adjacency list are sorted by increasing edge cost.

```
void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id);
    v.done = true;
    v.parent = parent;
    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;
        if (!w.done) {
            dfTrav(w, v);
        }
        e = e.next;
    }
}
```

```
dfTrav(Ptl, null)
  w = Pts
  dfTrav(Pts, Ptl)
    w = Ptl, Bos
    dfTrav(Bos, Pts)
      w = Wor
      dfTrav(Wor, Bos)
        w = Pro
        dfTrav(Pro, Wor)
          w = Wor, Bos, NY
          dfTrav(NY, Pro)
            w = Pro
            return
          no more neighbors
          return
        w = Bos, Con
        dfTrav(Con, Wor)
          …
```

Depth-First Spanning Tree



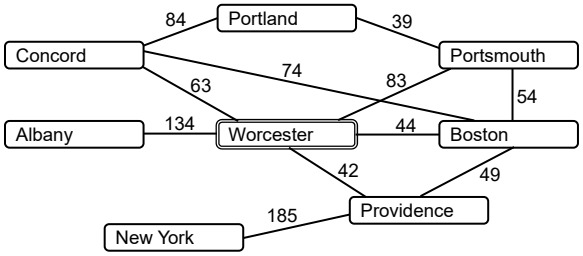


The edges obtained by following the `parent` references form a spanning tree with the origin of the traversal as its root.

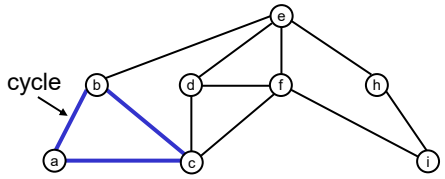
From any city, we can get to the origin by following the roads in the spanning tree.

Another Example: Depth-First Traversal from Worcester

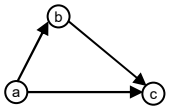
- In what order will the cities be visited?
- Which edges will be in the resulting spanning tree?



Checking for Cycles in an Undirected Graph



- To discover a cycle in an undirected graph, we can:
  - perform a depth-first traversal, marking the vertices as visited
  - if a visited vertex has a neighbor that is (1) not its parent, and (2) already marked as visited, there must be a cycle
- If no cycles found during the traversal, the graph is acyclic.
- This doesn't work for directed graphs:
  - c is a neighbor of both a and b
  - there is no cycle



Breadth-First Traversal

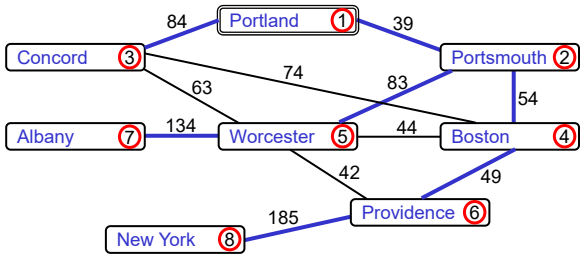
- Use a queue to store vertices we've seen but not yet visited:

```
private static void bfTrav(Vertex origin) {
    origin.encountered = true;
    origin.parent = null;
    Queue<Vertex> q = new LLQueue<Vertex>();
    q.insert(origin);

    while (!q.isEmpty()) {
        Vertex v = q.remove();
        System.out.println(v.id);          // Visit v.

        // Add v's unencountered neighbors to the queue.
        Edge e = v.edges;
        while (e != null) {
            Vertex w = e.end;
            if (!w.encountered) {
                w.encountered = true;
                w.parent = v;
                q.insert(w);
            }
            e = e.next;
        }
    }
}
```

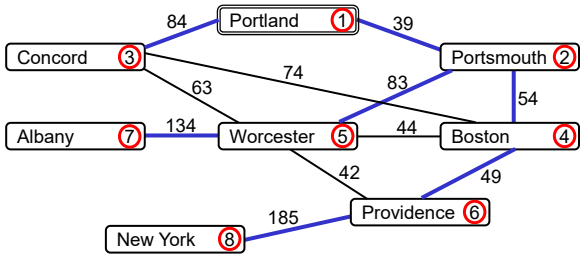
Example: Breadth-First Traversal from Portland



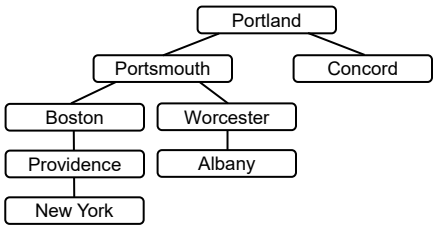
Evolution of the queue:

remove	insert	queue contents
	Portland	Portland
Portland	Portsmouth, Concord	Portsmouth, Concord
Portsmouth	Boston, Worcester	Concord, Boston, Worcester
Concord	*none*	Boston, Worcester
Boston	Providence	Worcester, Providence
Worcester	Albany	Providence, Albany
Providence	New York	Albany, New York
Albany	*none*	New York
New York	*none*	*empty*

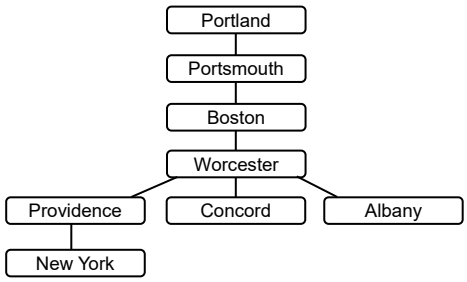
Breadth-First Spanning Tree



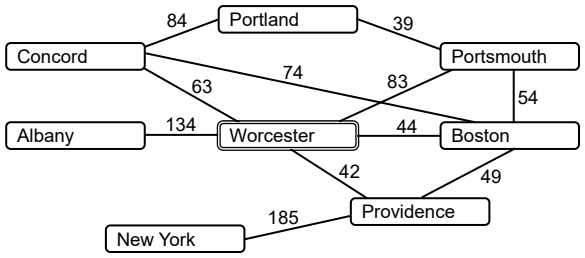
breadth-first spanning tree:



depth-first spanning tree:



Another Example: Breadth-First Traversal from Worcester



Evolution of the queue:

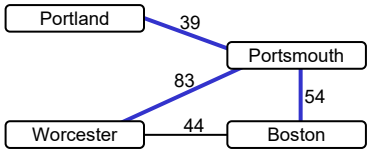
remove	insert	queue contents

Time Complexity of Graph Traversals

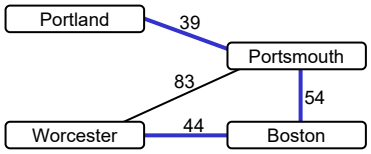
- let V = number of vertices in the graph
  E = number of edges
- If we use an adjacency matrix, a traversal requires $O(V^2)$ steps.
  - why?
- If we use adjacency lists, a traversal requires $O(V + E)$ steps.
  - visit each vertex once
  - traverse each vertex's adjacency list at most once
    - the total length of the adjacency lists is at most $2E = O(E)$
  - for a sparse graph, $O(V + E)$ is better than $O(V^2)$
  - for a dense graph, $E = O(V^2)$, so both representations are $O(V^2)$
- In the remaining notes, we'll assume an adjacency-list implementation.

Minimum Spanning Tree

- A *minimum spanning tree* (MST) has the smallest total cost among all possible spanning trees.
  - *example:*



one possible spanning tree
(total cost = 39 + 83 + 54 = 176)

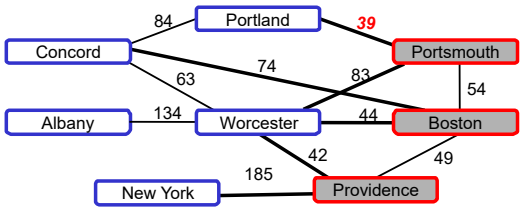


the minimal-cost spanning tree
(total cost = 39 + 54 + 44 = 137)

- If all edges have unique costs, there is only one MST.
  If some edges have the same cost, there may be more than one.
- Example applications:
  - determining the shortest highway system for a set of cities
  - calculating the smallest length of cable needed to connect a network of computers

Building a Minimum Spanning Tree

- *Claim:* If you divide the vertices into two disjoint subsets A and B, the lowest-cost edge (v_a, v_b) joining a vertex in A to a vertex in B must be part of the MST.



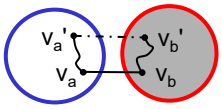
example:
subset A = unshaded
subset B = shaded

The 6 bold edges each join a vertex in A to a vertex in B.

The one with the lowest cost (Portland to Portsmouth) must be in the MST.

Proof by contradiction:

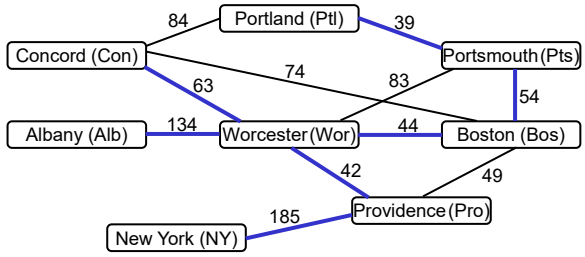
1. Assume we can create an MST (call it T) that doesn't include (v_a, v_b).
2. T must include a path from v_a to v_b, so it must include one of the other edges (v_a', v_b') that span A and B, such that (v_a', v_b') is part of the path from v_a to v_b.
3. Adding (v_a, v_b) to T introduces a cycle.
4. Removing (v_a', v_b') gives a spanning tree with a lower total cost, which contradicts the original assumption.



Prim's MST Algorithm

- Begin with the following subsets:
  - A = any one of the vertices
  - B = all of the other vertices
- Repeatedly do the following:
  - select the lowest-cost edge (v_a, v_b) connecting a vertex in A to a vertex in B
  - add (v_a, v_b) to the spanning tree
  - move vertex v_b from set B to set A
- Continue until set A contains all of the vertices.

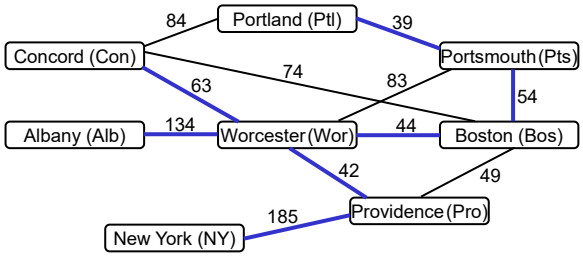
Example: Prim's Starting from Concord



- Tracing the algorithm:

edge added	set A	set B
	{Con}	{Alb, Bos, NY, Ptl, Pts, Pro, Wor}
(Con, Wor)	{Con, Wor}	{Alb, Bos, NY, Ptl, Pts, Pro}
(Wor, Pro)	{Con, Wor, Pro}	{Alb, Bos, NY, Ptl, Pts}
(Wor, Bos)	{Con, Wor, Pro, Bos}	{Alb, NY, Ptl, Pts}
(Bos, Pts)	{Con, Wor, Pro, Bos, Pts}	{Alb, NY, Ptl}
(Pts, Ptl)	{Con, Wor, Pro, Bos, Pts, Ptl}	{Alb, NY}
(Wor, Alb)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb}	{NY}
(Pro, NY)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb, NY}	{}

MST May Not Give Shortest Paths



- The MST is the spanning tree with the minimal *total* edge cost.
- It does not necessarily include the minimal cost path between a pair of vertices.
- Example: shortest path from Boston to Providence is along the single edge connecting them
  - that edge is not in the MST

Implementing Prim's Algorithm

- We use the `done` field to keep track of the sets.
  - if `v.done == true`, `v` is in set A
  - if `v.done == false`, `v` is in set B
- We repeatedly scan through the lists of vertices and edges to find the next edge to add.
  ➔ *O*(EV)
- We can do better!
  - use a heap-based priority queue to store the vertices in set B
  - priority of a vertex x = –1 * cost of the lowest-cost edge connecting x to a vertex in set A
    - why multiply by –1?
  - somewhat tricky: need to update the priorities over time

  ➔ *O*(E log V)

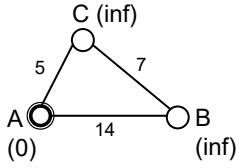
The Shortest-Path Problem

- It's often useful to know the shortest path from one vertex to another – i.e., the one with the minimal total cost
  - example application: routing traffic in the Internet
- For an *unweighted* graph, we can simply do the following:
  - start a breadth-first traversal from the origin, v
  - stop the traversal when you reach the other vertex, w
  - the path from v to w in the resulting (possibly partial) spanning tree is a shortest path
- A breadth-first traversal works for an unweighted graph because:
  - the shortest path is simply one with the fewest edges
  - a breadth-first traversal visits cities in order according to the number of edges they are from the origin.
- Why might this approach fail to work for a *weighted* graph?

Dijkstra's Algorithm

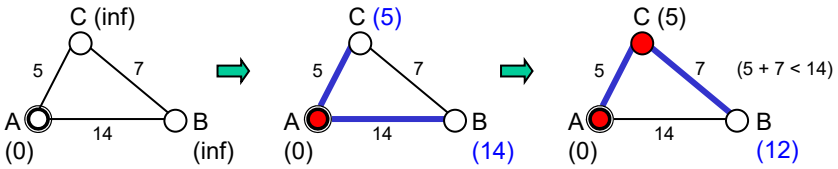
- One algorithm for solving the shortest-path problem for weighted graphs was developed by E.W. Dijkstra.
- It allows us to find the shortest path from a vertex v (the origin) to *all other vertices* that can be reached from v.
- Basic idea:
  - maintain estimates of the shortest paths from the origin to every vertex (along with their costs)
  - gradually refine these estimates as we traverse the graph
- Initial estimates:

	path	cost
the origin itself:	stay put!	0
all other vertices:	unknown	infinity



Dijkstra's Algorithm (cont.)

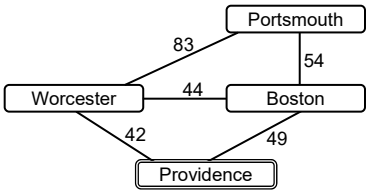
- We say that a vertex w is *finalized* if we have found the shortest path from v to w.
- We repeatedly do the following:
  - find the unfinalized vertex w with the lowest cost estimate
  - mark w as finalized (shown as a filled circle below)
  - examine each unfinalized neighbor x of w to see if there is a shorter path to x that passes through w
    - if there is, update the shortest-path estimate for x
- Example:



Another Example: Shortest Paths from Providence

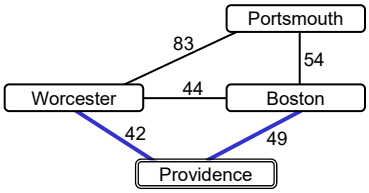
- Initial estimates:

Boston	infinity
Worcester	infinity
Portsmouth	infinity
Providence	***0***



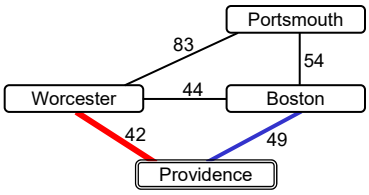
- Providence has the smallest unfinalized estimate, so we finalize it.
- We update our estimates for its neighbors:

Boston	*49* (< infinity)
Worcester	*42* (< infinity)
Portsmouth	infinity
Providence	0



Shortest Paths from Providence (cont.)

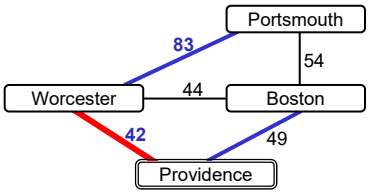
Boston	49
Worcester	***42***
Portsmouth	infinity
Providence	0



- Worcester has the smallest unfinalized estimate, so we finalize it.
  - any other route from Prov. to Worc. would need to go via Boston, and since (Prov → Worc) < (Prov → Bos), we can't do better.

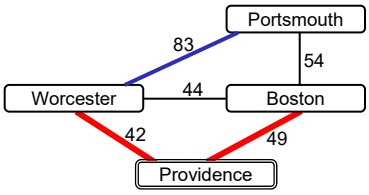
- We update our estimates for Worcester's unfinalized neighbors:

Boston	49 (no change)
Worcester	42
Portsmouth	*125* (42 + 83 < infinity)
Providence	0



Shortest Paths from Providence (cont.)

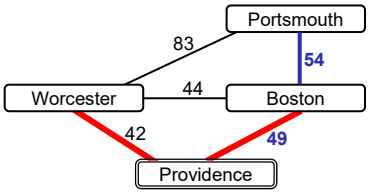
Boston	***49***
Worcester	42
Portsmouth	125
Providence	0



- Boston has the smallest unfinalized estimate, so we finalize it.
  - we'll see later why we can safely do this!

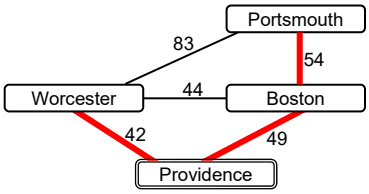
- We update our estimates for Boston's unfinalized neighbors:

Boston	49
Worcester	42
Portsmouth	*103* (49 + 54 < 125)
Providence	0



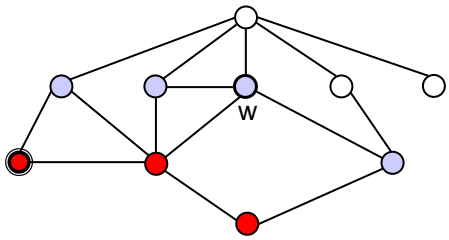
Shortest Paths from Providence (cont.)

Boston	49
Worcester	42
Portsmouth	***103***
Providence	0



- Only Portsmouth is left, so we finalize it.

Finalizing a Vertex



- origin
- other finalized vertices
- encountered but unfinalized (i.e., it has a non-infinite estimate)

- Let w be the unfinalized vertex with the smallest cost estimate. Why can we finalize w, before seeing the rest of the graph?
- We know that w's current estimate is for the shortest path to w that passes through only *finalized* vertices.
- Any shorter path to w would have to pass through one of the other encountered-but-unfinalized vertices, but they are all further away from the origin than w is!
  - their cost estimates may decrease in subsequent stages, but they can't drop below w's current estimate!

Pseudocode for Dijkstra's Algorithm

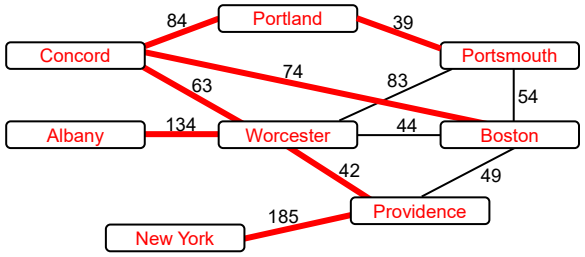
```
dijkstra(origin)
    origin.cost = 0
    for each other vertex v
        v.cost = infinity;

    while there are still unfinalized vertices with cost < infinity
        find the unfinalized vertex w with the minimal cost
        mark w as finalized

        for each unfinalized vertex x adjacent to w
            cost_via_w = w.cost + edge_cost(w, x)
            if (cost_via_w < x.cost)
                x.cost = cost_via_w
                x.parent = w
```

- At the conclusion of the algorithm, for each vertex v:
  - v.cost is the cost of the shortest path from the origin to v
  - if v.cost is infinity, there is no path from the origin to v
  - starting at v and following the parent references yields the shortest path

Example: Shortest Paths from Concord

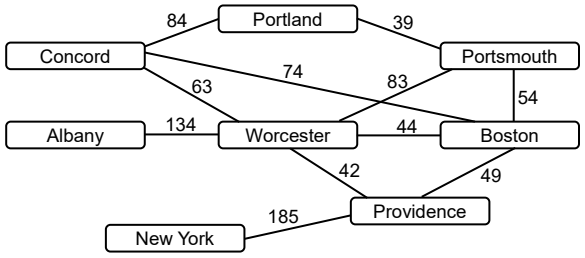


Evolution of the cost estimates (costs in bold have been finalized):

Albany	inf	inf	197	197	197	197	***197***	
Boston	inf	74	***74***					
Concord	***0***							
New York	inf	inf	inf	inf	inf	290	290	***290***
Portland	inf	84	84	***84***				
Portsmouth	inf	inf	146	128	123	***123***		
Providence	inf	inf	105	105	***105***			
Worcester	inf	***63***						

Note that the Portsmouth estimate was improved three times!

Another Example: Shortest Paths from Worcester



Evolution of the cost estimates (costs in bold have been finalized):

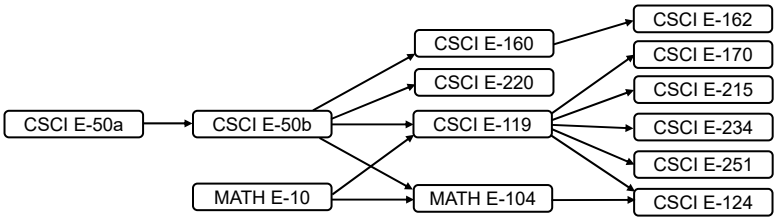
Albany								
Boston								
Concord								
New York								
Portland								
Portsmouth								
Providence								
Worcester								

Implementing Dijkstra's Algorithm

- Similar to the implementation of Prim's algorithm.
- Use a heap-based priority queue to store the unfinalized vertices.
  - priority = ?
- Need to update a vertex's priority whenever we update its shortest-path estimate.
- Time complexity = $O(E \log V)$

Topological Sort

- Used to order the vertices in a directed acyclic graph (a DAG).
- Topological order: an ordering of the vertices such that, if there is directed edge from a to b, a comes before b.
- Example application: ordering courses according to prerequisites



  - a directed edge from a to b indicates that a is a prereq of b
- There may be more than one topological ordering.

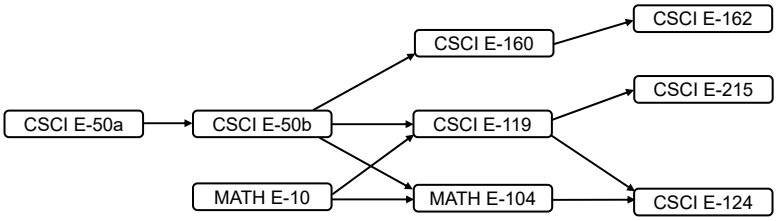
Topological Sort Algorithm

- A *successor* of a vertex v in a directed graph = a vertex w such that (v, w) is an edge in the graph (v•⟶•w)
- Basic idea: find vertices with no successors and work backward.
  - there must be at least one such vertex. why?
- Pseudocode for one possible approach:

```
topolSort
    S = a stack to hold the vertices as they are visited
    while there are still unvisited vertices
        find a vertex v with no unvisited successors
        mark v as visited
        S.push(v)
    return S
```

- Popping the vertices off the resulting stack gives one possible topological ordering.

Topological Sort Example

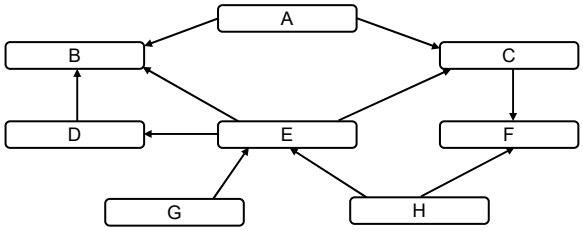


Evolution of the stack:

push	stack contents (top to bottom)
E-124	E-124
E-162	E-162, E-124
E-215	E-215, E-162, E-124
E-104	E-104, E-215, E-162, E-124
E-119	E-119, E-104, E-215, E-162, E-124
E-160	E-160, E-119, E-104, E-215, E-162, E-124
E-10	E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50b	E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50a	**E-50a, E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124**

one possible topological ordering

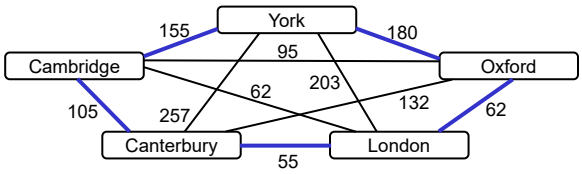
Another Topological Sort Example



Evolution of the stack:

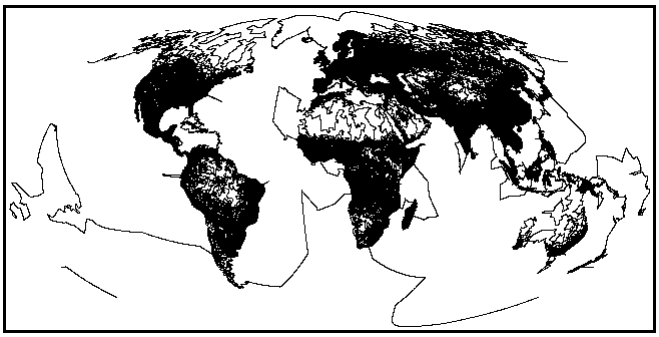
push	stack contents (top to bottom)

Traveling Salesperson Problem (TSP)



- A salesperson needs to travel to a number of cities to visit clients, and wants to do so as efficiently as possible.
- A *tour* is a path that:
  - begins at some starting vertex
  - passes through every other vertex *once and only once*
  - returns to the starting vertex
- TSP: find the tour with the lowest total cost

TSP for Santa Claus



A "world TSP" with 1,904,711 cities.

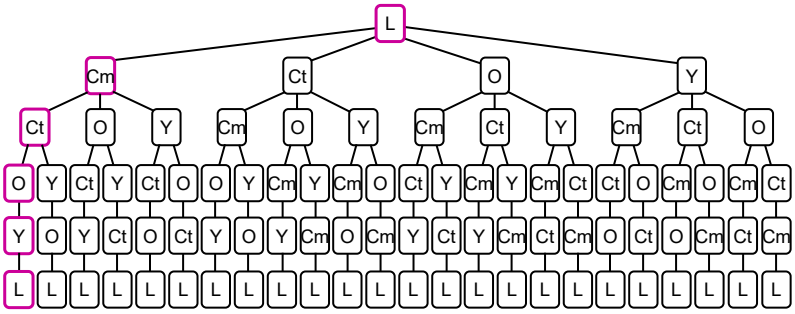
The figure at right shows a tour with a total cost of 7,516,353,779 meters – which is at most 0.068% longer than the optimal tour.

source: http://www.tsp.gatech.edu/world/pictures.html

- Other applications:
  - coin collection from phone booths
  - routes for school buses or garbage trucks
  - minimizing the movements of machines in automated manufacturing processes
  - many others

Solving a TSP: Brute-Force Approach

- Perform an exhaustive search of all possible tours.
  - represent the set of all possible tours as a tree



- The leaf nodes correspond to possible solutions.
  - for n cities, there are (n – 1)! leaf nodes in the tree.
  - half are redundant (e.g., L-Cm-Ct-O-Y-L = L-Y-O-Ct-Cm-L)
- Problem: exhaustive search is intractable for all but small n.
  - example: when n = 14, ((n – 1)!) / 2 = over 3 billion

Solving a TSP: Informed Search

- Focus on the most promising paths through the tree of possible tours.
  - use a function that estimates how good a given path is
- Better than brute force, but still exponential space and time.

Algorithm Analysis Revisited

- Recall that we can group algorithms into classes (n = problem size):

name	example expressions	big-O notation
constant time	$1, 7, 10$	$O(1)$
logarithmic time	$3\log_{10}n, \log_2 n + 5$	$O(\log n)$
linear time	$5n, 10n - 2\log_2 n$	$O(n)$
n log n time	$4n \log_2 n, n \log_2 n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n, n^2 - 1$	$O(n^2)$
n^c ($c > 2$)	$n^3 - 5n, 2n^5 + 5n^2$	$O(n^c)$
exponential time	$2^n, 5e^n + 2n^2$	$O(c^n)$
factorial time	$(n-1)!/2, 3n!$	$O(n!)$

- Algorithms that fall into one of the classes above the dotted line are referred to as *polynomial-time* algorithms.
- The term *exponential-time algorithm* is sometimes used to include *all* algorithms that fall below the dotted line.
  - algorithms whose running time grows as fast or faster than c^n

Classifying Problems

- Problems that can be solved using a polynomial-time algorithm are considered “easy” problems.
  - we can solve large problem instances in a reasonable amount of time
- Problems that don’t have a polynomial-time solution algorithm are considered “hard” or "intractable" problems.
  - they can only be solved exactly for small values of n
- Increasing the CPU speed doesn't help much for intractable problems:

	CPU 1	CPU 2 (1000x faster)
max problem size for $O(n)$ alg:	N	1000N
$O(n^2)$ alg:	N	31.6 N
$O(2^n)$ alg:	N	**N + 9.97**

Dealing With Intractable Problems

- When faced with an intractable problem, we resort to techniques that quickly find solutions that are "good enough".
- Such techniques are often referred to as *heuristic* techniques.
  - heuristic = rule of thumb
  - there's no guarantee these techniques will produce the optimal solution, but they typically work well

Take-Home Lessons

- Object-oriented programming allows us to capture the abstractions in the programs that we write.
  - creates reusable building blocks
  - key concepts: encapsulation, inheritance, polymorphism
- Abstract data types allow us to organize and manipulate collections of data.
  - a given ADT can be implemented in different ways
  - fundamental building blocks: arrays, linked nodes
- Efficiency matters when dealing with large collections of data.
  - some solutions can be *much* faster or more space efficient
  - what's the best data structure/algorithm for *your* workload?
    - example: sorting an almost sorted collection

Take-Home Lessons (cont.)

- Use the tools in your toolbox!
  - interfaces, generic data structures
  - lists/stacks/queues, trees, heaps, hash tables
  - recursion, recursive backtracking, divide-and-conquer
- Use built-in/provided collections/interfaces:
  - `java.util.ArrayList<T>` (implements `List<T>`)
  - `java.util.LinkedList<T>` (implements `List<T>` and `Queue<T>`)
  - `java.util.Stack<T>`
  - `java.util.TreeMap<K, V>` (a balanced search tree)
  - `java.util.HashMap<K, V>` (a hash table)

    } implement `Map<K, V>`
  - `java.util.PriorityQueue<T>` (a heap)
- But use them intelligently!
  - ex: `LinkedList` maintains a reference to the last node in the list
  - `list.add(item, n)` will add `item` to the end in O(n) time
  - `list.addLast(item)` will add `item` to the end in O(1) time!