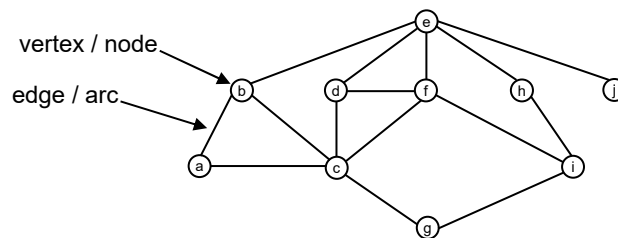


Graphs

Computer Science E-22
Harvard Extension School

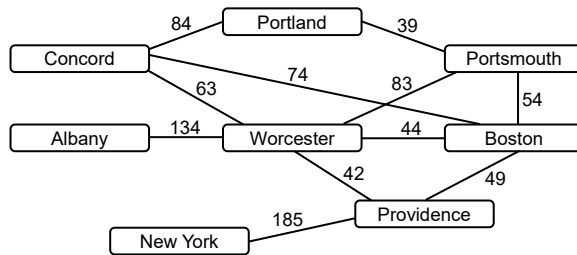
David G. Sullivan, Ph.D.

What is a Graph?



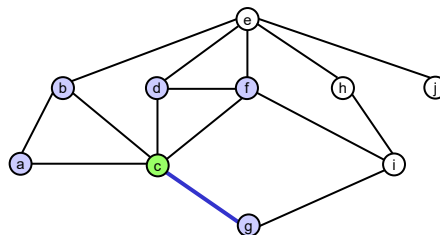
- A graph consists of:
 - a set of *vertices* (also known as *nodes*)
 - a set of *edges* (also known as *arcs*), each of which connects a pair of vertices

Example: A Highway Graph



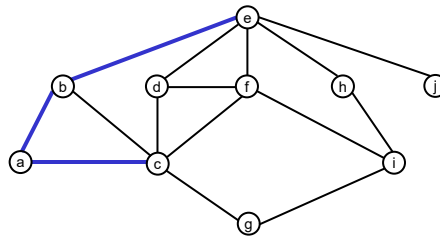
- Vertices represent cities.
- Edges represent highways.
- This is a *weighted* graph, with a *cost* associated with each edge.
 - in this example, the costs denote mileage
- We'll use graph algorithms to answer questions like "What is the shortest route from Portland to Providence?"

Relationships Among Vertices

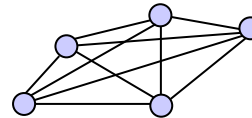
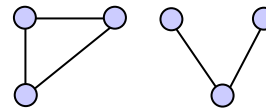


- Two vertices are *adjacent* if they are connected by a single edge.
 - ex: c and g are adjacent, but c and i are not
- The collection of vertices that are adjacent to a vertex v are referred to as v's *neighbors*.
 - ex: c's neighbors are a, b, d, f, and g

Paths in a Graph

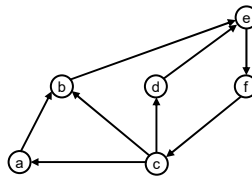


- A *path* is a sequence of edges that connects two vertices.
- A graph is *connected* if there is a path between any two vertices.
 - ex: the six vertices at right are part of a graph that is *not* connected
- A graph is *complete* if there is an edge between every pair of vertices.
 - ex: the graph at right *is* complete



Directed Graphs

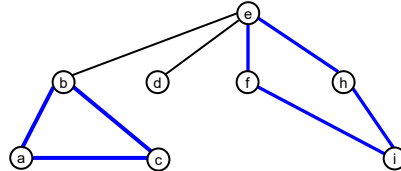
- A *directed* graph has a direction associated with each edge, which is depicted using an arrow:



- Edges in a directed graph are often represented as ordered pairs of the form (start vertex, end vertex).
 - ex: (a, b) is an edge in the graph above, but (b, a) is not.
- In a path in a directed graph, the end vertex of edge i must be the same as the start vertex of edge $i + 1$.
 - ex: $\{(a, b), (b, e), (e, f)\}$ is a valid path.
 $\{(a, b), (c, b), (c, a)\}$ is not.

Cycles in a Graph

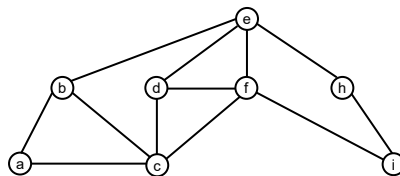
- A *cycle* is a path that:
 - leaves a given vertex using one edge
 - returns to that same vertex using a different edge
- Examples: the highlighted paths below



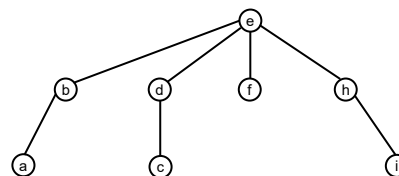
- An *acyclic* graph has no cycles.

Trees vs. Graphs

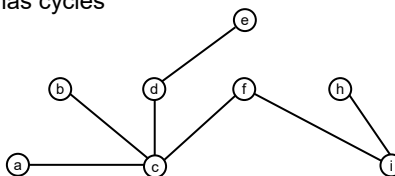
- A tree is a special type of graph.
 - connected, undirected, and acyclic
 - we usually single out one of the vertices to be the root, but graph theory does *not* require this



a graph that is *not* a tree,
because it has cycles



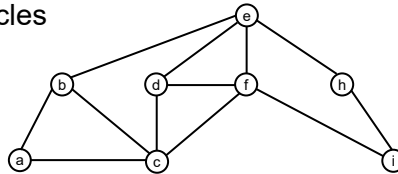
a tree using the same nodes



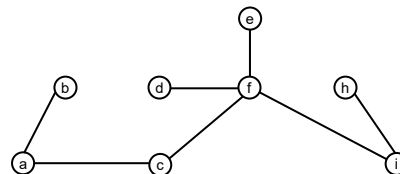
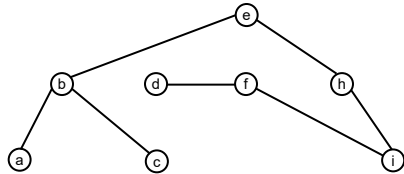
another tree using the same nodes

Spanning Trees

- A spanning tree is a subset of a connected graph that contains:
 - all of the vertices
 - a subset of the edges that form a tree
- Recall this graph with cycles from the previous slide:



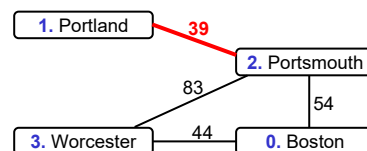
- The trees on that slide were spanning trees for this graph. Here are two others:



Representing a Graph: Option 1

- Use an *adjacency matrix* – a two-dimensional array in which element $[r][c]$ = the cost of going from vertex r to vertex c
- Example:

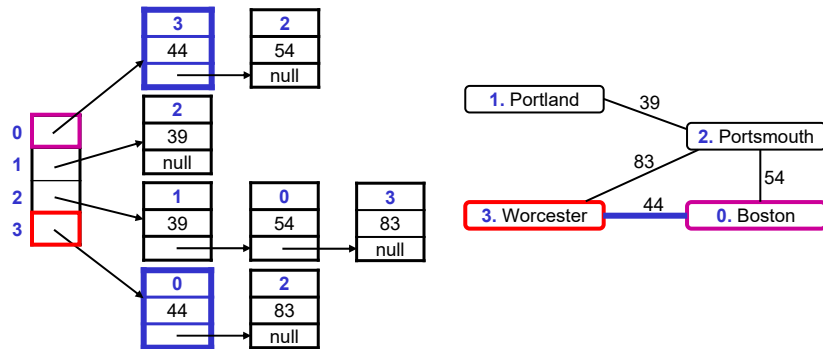
	0	1	2	3
0			54	44
1			39	
2	54	39		83
3	44		83	



- Use a special value to indicate there's no edge from r to c
 - shown as a shaded cell above
 - can't use 0, because an edge may have an actual cost of 0
- This representation:
 - wastes memory if a graph is *sparse* (few edges per vertex)
 - is memory-efficient if a graph is *dense* (many edges per vertex)

Representing a Graph: Option 2

- Use one *adjacency list* for each vertex.
 - a linked list with info on the edges coming from that vertex



- This representation uses less memory if a graph is sparse.
- It uses more memory if a graph is dense.
 - because of the references linking the nodes

Graph Class

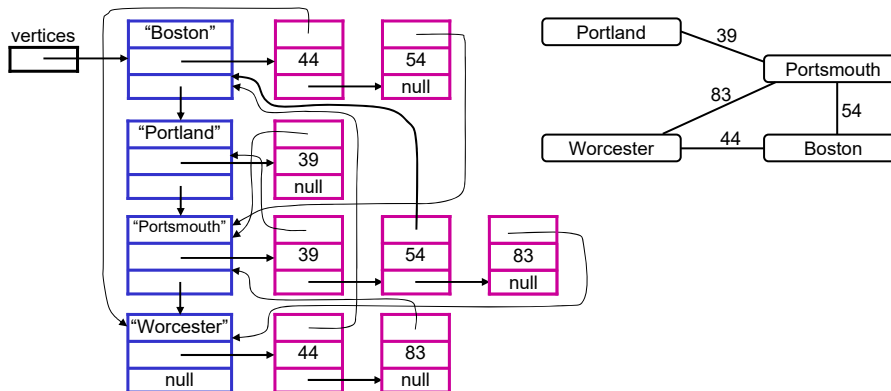
```
public class Graph {
    private class Vertex {
        private String id;
        private Edge edges;           // adjacency list
        private Vertex next;
        private boolean encountered;
        private boolean done;
        private Vertex parent;
        private double cost;
        ...
    }

    private class Edge {
        private Vertex start;
        private Vertex end;
        private double cost;
        private Edge next;
        ...
    }

    private Vertex vertices;
    ...
}
```

The highlighted fields
are shown in the diagram
on the previous page.

Our Graph Representation



- Each Vertex object (shown in blue) stores info. about a vertex.
 - including an adjacency list of Edge objects (the purple ones)
- A Graph object has a single field called `vertices`
 - a reference to a linked list of Vertex objects
 - a linked list of linked lists!

Traversing a Graph

- Traversing a graph involves starting at some vertex and visiting all vertices that can be reached from that vertex.
 - visiting a vertex = processing its data in some way
 - if the graph is connected, all of its vertices will be visited
- We will consider two types of traversals:
 - **depth-first:** proceed as far as possible along a given path before backing up
 - **breadth-first:** visit a vertex
visit all of its neighbors
visit all unvisited vertices 2 edges away
visit all unvisited vertices 3 edges away, etc.

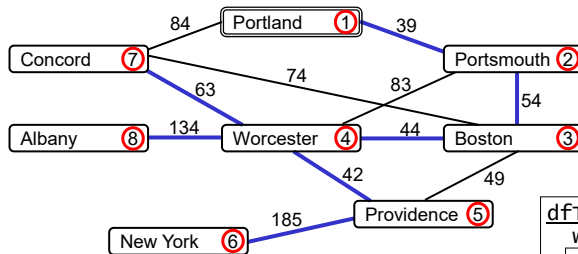
Depth-First Traversal

- Visit a vertex, then make recursive calls on all of its yet-to-be-visited neighbors:

```
private static void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id); // visit v
    v.done = true;
    v.parent = parent;       // record where we came from

    // walk down v's adjacency list
    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;    // consider each neighbor w
        if (!w.done) {       // if w has not been visited
            dfTrav(w, v);
        }
        e = e.next;
    }
}
```

Example: Depth-First Traversal from Portland

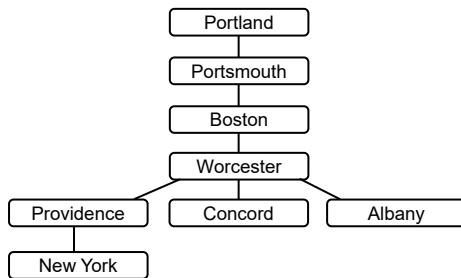
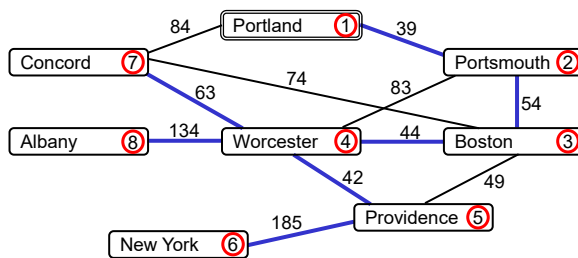


For the examples, we'll assume that the edges in each vertex's adjacency list are sorted by increasing edge cost.

```
void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id);
    v.done = true;
    v.parent = parent;
    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;
        if (!w.done) {
            dfTrav(w, v);
        }
        e = e.next;
    }
}
```

```
dfTrav(Ptl, null)
w = Pts
dfTrav(Pts, Ptl)
w = Ptl, Bos
dfTrav(Bos, Pts)
w = Wor
dfTrav(Wor, Bos)
w = Pro
dfTrav(Pro, wor)
w = Wor, Bos, NY
dfTrav(NY, Pro)
w = Pro
return
no more neighbors
return
w = Bos, Con
dfTrav(Con, wor)
...
```

Depth-First Spanning Tree

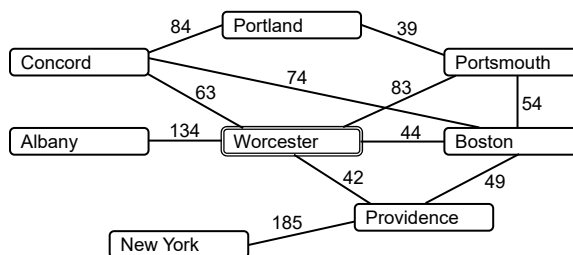


The edges obtained by following the parent references form a spanning tree with the origin of the traversal as its root.

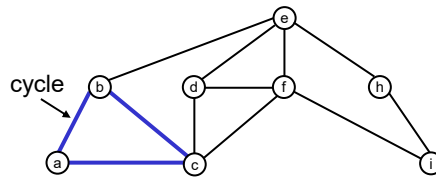
From any city, we can get to the origin by following the roads in the spanning tree.

Another Example: Depth-First Traversal from Worcester

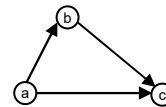
- In what order will the cities be visited?
- Which edges will be in the resulting spanning tree?



Checking for Cycles in an Undirected Graph



- To discover a cycle in an undirected graph, we can:
 - perform a depth-first traversal, marking the vertices as visited
 - if a visited vertex has a neighbor that is (1) not its parent, and (2) already marked as visited, there must be a cycle
- If no cycles found during the traversal, the graph is acyclic.
- This doesn't work for directed graphs:
 - c is a neighbor of both a and b
 - there is no cycle

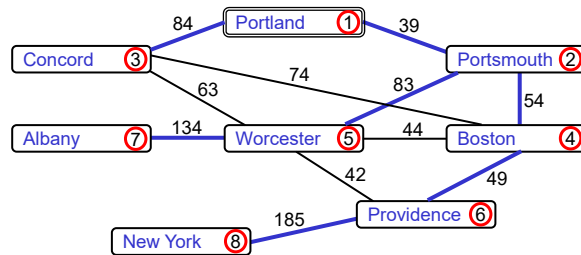


Breadth-First Traversal

- Use a queue to store vertices we've seen but not yet visited:

```
private static void bfTrav(Vertex origin) {
    origin.encountered = true;
    origin.parent = null;
    Queue<Vertex> q = new LLQueue<Vertex>();
    q.insert(origin);
    while (!q.isEmpty()) {
        Vertex v = q.remove();
        System.out.println(v.id);           // visit v.
        // Add v's unencountered neighbors to the queue.
        Edge e = v.edges;
        while (e != null) {
            Vertex w = e.end;
            if (!w.encountered) {
                w.encountered = true;
                w.parent = v;
                q.insert(w);
            }
            e = e.next;
        }
    }
}
```

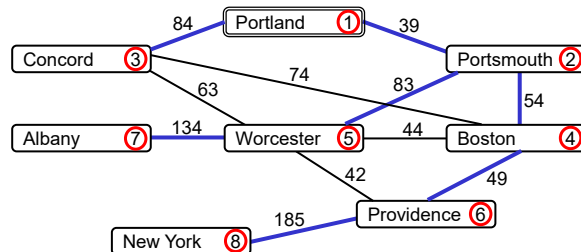
Example: Breadth-First Traversal from Portland



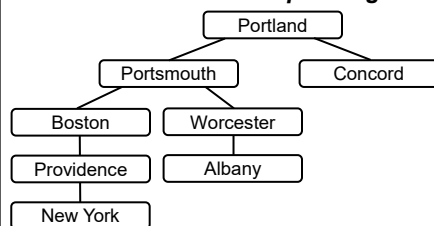
Evolution of the queue:

<u>remove</u>	<u>insert</u>	<u>queue contents</u>
	Portland	Portland
Portland	Portsmouth, Concord	Portsmouth, Concord
Portsmouth	Boston, Worcester	Concord, Boston, Worcester
Concord	<i>none</i>	Boston, Worcester
Boston	Providence	Worcester, Providence
Worcester	Albany	Providence, Albany
Providence	New York	Albany, New York
Albany	<i>none</i>	New York
New York	<i>none</i>	<i>empty</i>

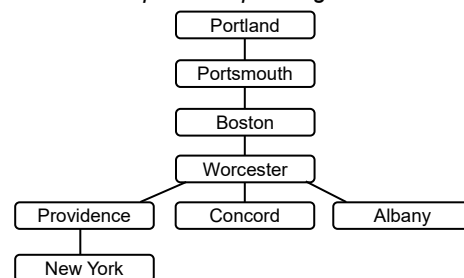
Breadth-First Spanning Tree



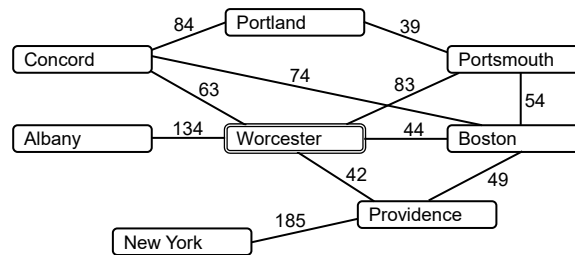
breadth-first spanning tree:



depth-first spanning tree:



Another Example: Breadth-First Traversal from Worcester



Evolution of the queue:

remove

insert

queue contents

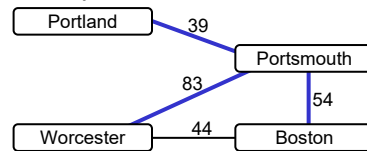
Time Complexity of Graph Traversals

- let V = number of vertices in the graph
 E = number of edges
- If we use an adjacency matrix, a traversal requires $O(V^2)$ steps.
 - why?
- If we use adjacency lists, a traversal requires $O(V + E)$ steps.
 - visit each vertex once
 - traverse each vertex's adjacency list at most once
 - the total length of the adjacency lists is at most $2E = O(E)$
 - for a sparse graph, $O(V + E)$ is better than $O(V^2)$
 - for a dense graph, $E = O(V^2)$, so both representations are $O(V^2)$
- In the remaining notes, we'll assume an adjacency-list implementation.

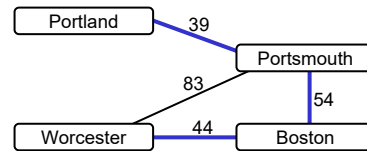
Minimum Spanning Tree

- A *minimum spanning tree* (MST) has the smallest total cost among all possible spanning trees.

- example:*



one possible spanning tree
(total cost = 39 + 83 + 54 = 176)

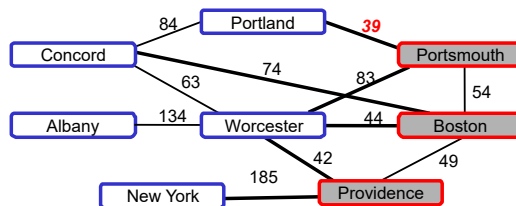


the minimal-cost spanning tree
(total cost = 39 + 54 + 44 = 137)

- If all edges have unique costs, there is only one MST.
If some edges have the same cost, there may be more than one.
- Example applications:
 - determining the shortest highway system for a set of cities
 - calculating the smallest length of cable needed to connect a network of computers

Building a Minimum Spanning Tree

- Claim:* If you divide the vertices into two disjoint subsets A and B, the lowest-cost edge (v_a, v_b) joining a vertex in A to a vertex in B must be part of the MST.



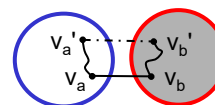
example:
subset A = unshaded
subset B = shaded

The 6 bold edges each join a vertex in A to a vertex in B.

The one with the lowest cost (Portland to Portsmouth) must be in the MST.

Proof by contradiction:

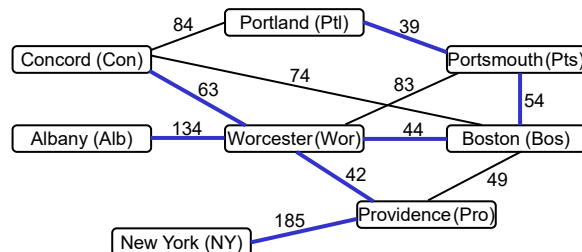
- Assume we can create an MST (call it T) that doesn't include (v_a, v_b) .
- T must include a path from v_a to v_b , so it must include one of the other edges (v_a', v_b') that span A and B, such that (v_a', v_b') is part of the path from v_a to v_b .
- Adding (v_a, v_b) to T introduces a cycle.
- Removing (v_a', v_b') gives a spanning tree with a lower total cost, which contradicts the original assumption.



Prim's MST Algorithm

- Begin with the following subsets:
 - A = any one of the vertices
 - B = all of the other vertices
- Repeatedly do the following:
 - select the lowest-cost edge (v_a, v_b) connecting a vertex in A to a vertex in B
 - add (v_a, v_b) to the spanning tree
 - move vertex v_b from set B to set A
- Continue until set A contains all of the vertices.

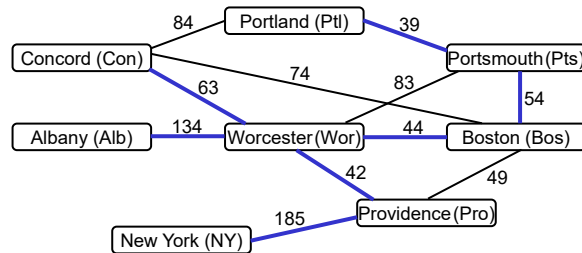
Example: Prim's Starting from Concord



- Tracing the algorithm:

edge added	set A	set B
	{Con}	{Alb, Bos, NY, Ptl, Pts, Pro, Wor}
(Con, Wor)	{Con, Wor}	{Alb, Bos, NY, Ptl, Pts, Pro}
(Wor, Pro)	{Con, Wor, Pro}	{Alb, Bos, NY, Ptl, Pts}
(Wor, Bos)	{Con, Wor, Pro, Bos}	{Alb, NY, Ptl, Pts}
(Bos, Pts)	{Con, Wor, Pro, Bos, Pts}	{Alb, NY, Ptl}
(Pts, Ptl)	{Con, Wor, Pro, Bos, Pts, Ptl}	{Alb, NY}
(Wor, Alb)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb}	{NY}
(Pro, NY)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb, NY}	{}

MST May Not Give Shortest Paths



- The MST is the spanning tree with the minimal *total* edge cost.
- It does not necessarily include the minimal cost path between a pair of vertices.
- Example: shortest path from Boston to Providence is along the single edge connecting them
 - that edge is not in the MST

Implementing Prim's Algorithm

- We use the done field to keep track of the sets.
 - if `v.done == true`, `v` is in set A
 - if `v.done == false`, `v` is in set B
- We repeatedly scan through the lists of vertices and edges to find the next edge to add.
 - ➔ $O(EV)$
- We can do better!
 - use a heap-based priority queue to store the vertices in set B
 - priority of a vertex `x` = $-1 \times \text{cost of the lowest-cost edge connecting } x \text{ to a vertex in set A}$
 - why multiply by -1 ?
 - somewhat tricky: need to update the priorities over time
 - ➔ $O(E \log V)$

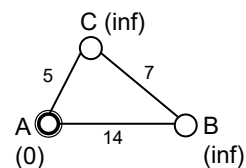
The Shortest-Path Problem

- It's often useful to know the shortest path from one vertex to another – i.e., the one with the minimal total cost
 - example application: routing traffic in the Internet
- For an *unweighted* graph, we can simply do the following:
 - start a breadth-first traversal from the origin, v
 - stop the traversal when you reach the other vertex, w
 - the path from v to w in the resulting (possibly partial) spanning tree is a shortest path
- A breadth-first traversal works for an unweighted graph because:
 - the shortest path is simply one with the fewest edges
 - a breadth-first traversal visits cities in order according to the number of edges they are from the origin.
- Why might this approach fail to work for a *weighted* graph?

Dijkstra's Algorithm

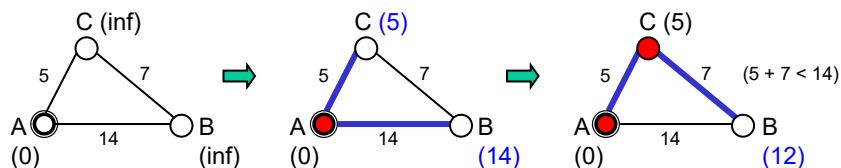
- One algorithm for solving the shortest-path problem for weighted graphs was developed by E.W. Dijkstra.
- It allows us to find the shortest path from a vertex v (the origin) to *all other vertices* that can be reached from v .
- Basic idea:
 - maintain estimates of the shortest paths from the origin to every vertex (along with their costs)
 - gradually refine these estimates as we traverse the graph
- Initial estimates:

	<u>path</u>	<u>cost</u>
the origin itself:	stay put!	0
all other vertices:	unknown	infinity



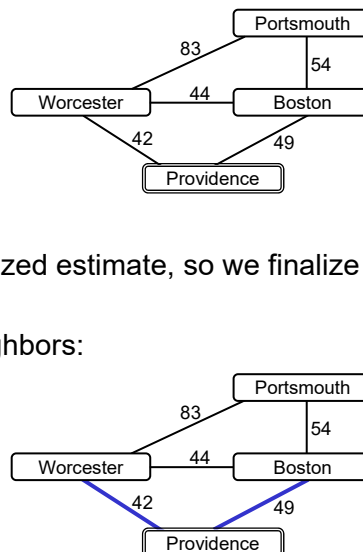
Dijkstra's Algorithm (cont.)

- We say that a vertex w is *finalized* if we have found the shortest path from v to w .
- We repeatedly do the following:
 - find the unfinalized vertex w with the lowest cost estimate
 - mark w as finalized (shown as a filled circle below)
 - examine each unfinalized neighbor x of w to see if there is a shorter path to x that passes through w
 - if there is, update the shortest-path estimate for x
- Example:



Another Example: Shortest Paths from Providence

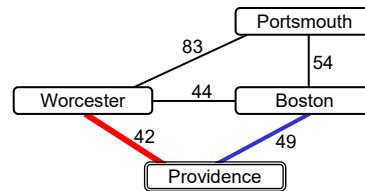
- Initial estimates:
- | | |
|------------|----------|
| Boston | infinity |
| Worcester | infinity |
| Portsmouth | infinity |
| Providence | 0 |
- Providence has the smallest unfinalized estimate, so we finalize it.
 - We update our estimates for its neighbors:



Boston	49 (< infinity)
Worcester	42 (< infinity)
Portsmouth	infinity
Providence	0

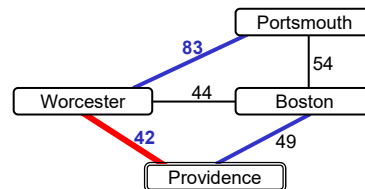
Shortest Paths from Providence (cont.)

Boston	49
Worcester	42
Portsmouth	infinity
Providence	0



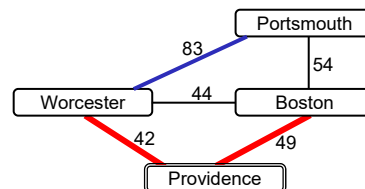
- Worcester has the smallest unfinalized estimate, so we finalize it.
 - any other route from Prov. to Worc. would need to go via Boston, and since $(\text{Prov} \rightarrow \text{Worc}) < (\text{Prov} \rightarrow \text{Bos})$, we can't do better.
- We update our estimates for Worcester's unfinalized neighbors:

Boston	49 (no change)
Worcester	42
Portsmouth	125 ($42 + 83 < \text{infinity}$)
Providence	0



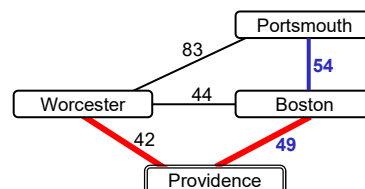
Shortest Paths from Providence (cont.)

Boston	49
Worcester	42
Portsmouth	125
Providence	0



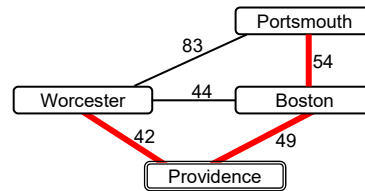
- Boston has the smallest unfinalized estimate, so we finalize it.
 - we'll see later why we can safely do this!
- We update our estimates for Boston's unfinalized neighbors:

Boston	49
Worcester	42
Portsmouth	103 ($49 + 54 < 125$)
Providence	0



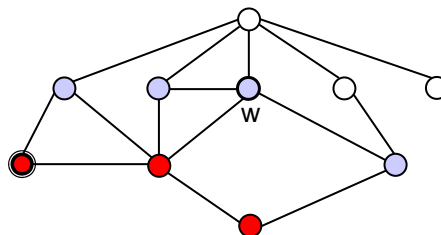
Shortest Paths from Providence (cont.)

Boston	49
Worcester	42
Portsmouth	103
Providence	0



- Only Portsmouth is left, so we finalize it.

Finalizing a Vertex



- origin
- other finalized vertices
- encountered but unfinalized
(i.e., it has a non-infinite estimate)

- Let w be the unfinalized vertex with the smallest cost estimate. Why can we finalize w , before seeing the rest of the graph?
- We know that w 's current estimate is for the shortest path to w that passes through only *finalized* vertices.
- Any shorter path to w would have to pass through one of the other encountered-but-unfinalized vertices, but they are all further away from the origin than w is!
 - their cost estimates may decrease in subsequent stages, but they can't drop below w 's current estimate!

Pseudocode for Dijkstra's Algorithm

```

dijkstra(origin)
  origin.cost = 0
  for each other vertex v
    v.cost = infinity;

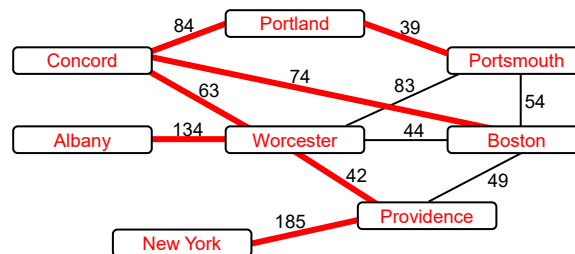
  while there are still unfinalized vertices with cost < infinity
    find the unfinalized vertex w with the minimal cost
    mark w as finalized

    for each unfinalized vertex x adjacent to w
      cost_via_w = w.cost + edge_cost(w, x)
      if (cost_via_w < x.cost)
        x.cost = cost_via_w
        x.parent = w

```

- At the conclusion of the algorithm, for each vertex v:
 - v.cost is the cost of the shortest path from the origin to v
 - if v.cost is infinity, there is no path from the origin to v
 - starting at v and following the parent references yields the shortest path

Example: Shortest Paths from Concord

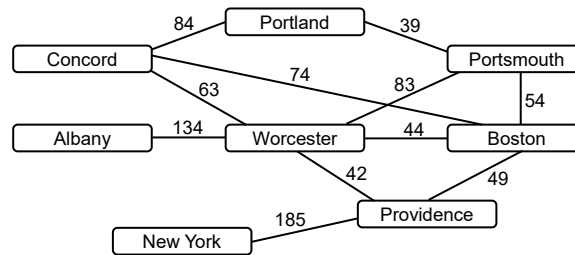


Evolution of the cost estimates (costs in bold have been finalized):

Albany	inf	inf	197	197	197	197	197	
Boston	inf	74	74					
Concord	0							
New York	inf	inf	inf	inf	inf	290	290	290
Portland	inf	84	84	84				
Portsmouth	inf	inf	146	128	123	123		
Providence	inf	inf	105	105	105			
Worcester	inf	63						

Note that the Portsmouth estimate was improved three times!

Another Example: Shortest Paths from Worcester



Evolution of the cost estimates (costs in bold have been finalized):

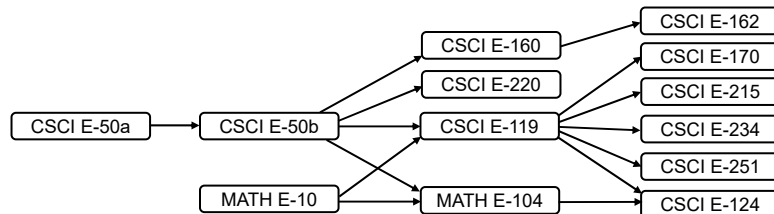
Albany									
Boston									
Concord									
New York									
Portland									
Portsmouth									
Providence									
Worcester									

Implementing Dijkstra's Algorithm

- Similar to the implementation of Prim's algorithm.
- Use a heap-based priority queue to store the unfinalized vertices.
 - priority = ?
- Need to update a vertex's priority whenever we update its shortest-path estimate.
- Time complexity = $O(E \log V)$

Topological Sort

- Used to order the vertices in a directed acyclic graph (a DAG).
- Topological order: an ordering of the vertices such that, if there is directed edge from a to b, a comes before b.
- Example application: ordering courses according to prerequisites



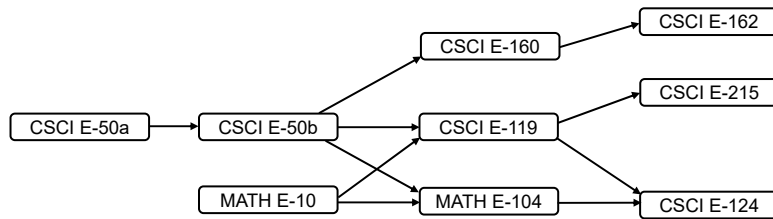
- a directed edge from a to b indicates that a is a prereq of b
- There may be more than one topological ordering.

Topological Sort Algorithm

- A *successor* of a vertex v in a directed graph = a vertex w such that (v, w) is an edge in the graph ($v \rightarrow w$)
- Basic idea: find vertices with no successors and work backward.
 - there must be at least one such vertex. why?
- Pseudocode for one possible approach:

```
topoSort
  S = a stack to hold the vertices as they are visited
  while there are still unvisited vertices
    find a vertex v with no unvisited successors
    mark v as visited
    S.push(v)
  return S
```
- Popping the vertices off the resulting stack gives one possible topological ordering.

Topological Sort Example

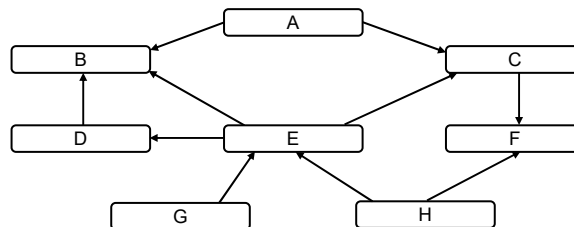


Evolution of the stack:

<u>push</u>	<u>stack contents (top to bottom)</u>
E-124	E-124
E-162	E-162, E-124
E-215	E-215, E-162, E-124
E-104	E-104, E-215, E-162, E-124
E-119	E-119, E-104, E-215, E-162, E-124
E-160	E-160, E-119, E-104, E-215, E-162, E-124
E-10	E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50b	E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50a	E-50a, E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124

one possible topological ordering

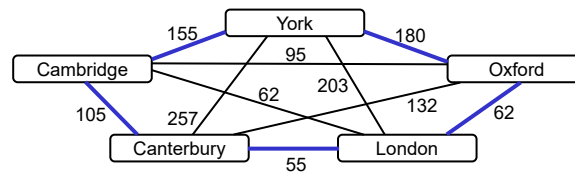
Another Topological Sort Example



Evolution of the stack:

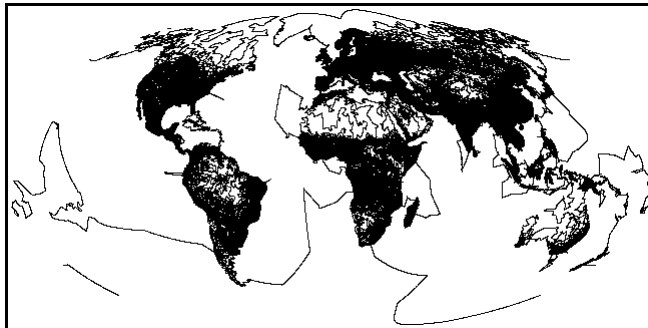
<u>push</u>	<u>stack contents (top to bottom)</u>
-------------	---------------------------------------

Traveling Salesperson Problem (TSP)



- A salesperson needs to travel to a number of cities to visit clients, and wants to do so as efficiently as possible.
- A *tour* is a path that:
 - begins at some starting vertex
 - passes through every other vertex *once and only once*
 - returns to the starting vertex
- TSP: find the tour with the lowest total cost

TSP for Santa Claus



source: <http://www.tsp.gatech.edu/world/pictures.html>

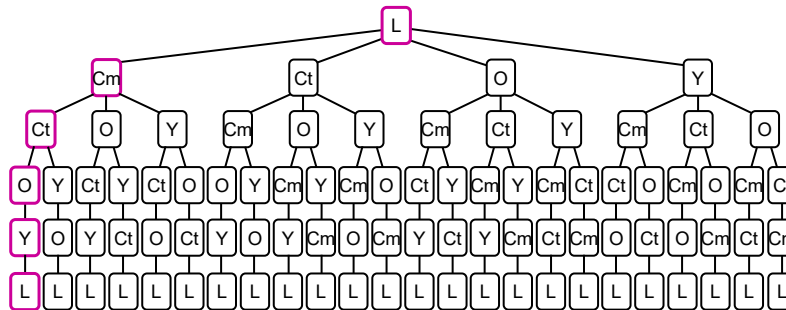
A "world TSP" with 1,904,711 cities.

The figure at right shows a tour with a total cost of 7,516,353,779 meters – which is at most 0.068% longer than the optimal tour.

- Other applications:
 - coin collection from phone booths
 - routes for school buses or garbage trucks
 - minimizing the movements of machines in automated manufacturing processes
 - many others

Solving a TSP: Brute-Force Approach

- Perform an exhaustive search of all possible tours.
 - represent the set of all possible tours as a tree



- The leaf nodes correspond to possible solutions.
 - for n cities, there are $(n - 1)!$ leaf nodes in the tree.
 - half are redundant (e.g., L-Cm-Ct-O-Y-L = L-Y-O-Ct-Cm-L)
- Problem: exhaustive search is intractable for all but small n .
 - example: when $n = 14$, $((n - 1)!)/2$ is over 3 billion

Solving a TSP: Informed Search

- Focus on the most promising paths through the tree of possible tours.
 - use a function that estimates how good a given path is
- Better than brute force, but still exponential space and time.

Algorithm Analysis Revisited

- Recall that we can group algorithms into classes (n = problem size):

<u>name</u>	<u>example expressions</u>	<u>big-O notation</u>
constant time	1, 7, 10	$O(1)$
logarithmic time	$3\log_{10}n$, $\log_2n + 5$	$O(\log n)$
linear time	$5n$, $10n - 2\log_2n$	$O(n)$
$n \log n$ time	$4n \log_2n$, $n \log_2n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n$, $n^2 - 1$	$O(n^2)$
n^c ($c > 2$)	$n^3 - 5n$, $2n^5 + 5n^2$	$O(n^c)$
exponential time	2^n , $5e^n + 2n^2$	$O(c^n)$
factorial time	$(n - 1)!/2$, $3n!$	$O(n!)$

- Algorithms that fall into one of the classes above the dotted line are referred to as *polynomial-time* algorithms.
- The term *exponential-time algorithm* is sometimes used to include *all* algorithms that fall below the dotted line.
 - algorithms whose running time grows as fast or faster than c^n

Classifying Problems

- Problems that can be solved using a polynomial-time algorithm are considered “easy” problems.
 - we can solve large problem instances in a reasonable amount of time
- Problems that don't have a polynomial-time solution algorithm are considered “hard” or “intractable” problems.
 - they can only be solved exactly for small values of n
- Increasing the CPU speed doesn't help much for intractable problems:

	<u>CPU 1</u>	<u>CPU 2</u> <u>(1000x faster)</u>
max problem size for $O(n)$ alg:	N	$1000N$
$O(n^2)$ alg:	N	$31.6 N$
$O(2^n)$ alg:	N	$N + 9.97$

Dealing With Intractable Problems

- When faced with an intractable problem, we resort to techniques that quickly find solutions that are "good enough".
- Such techniques are often referred to as *heuristic* techniques.
 - heuristic = rule of thumb
 - there's no guarantee these techniques will produce the optimal solution, but they typically work well

Take-Home Lessons

- Object-oriented programming allows us to capture the abstractions in the programs that we write.
 - creates reusable building blocks
 - key concepts: encapsulation, inheritance, polymorphism
- Abstract data types allow us to organize and manipulate collections of data.
 - a given ADT can be implemented in different ways
 - fundamental building blocks: arrays, linked nodes
- Efficiency matters when dealing with large collections of data.
 - some solutions can be *much* faster or more space efficient
 - what's the best data structure/algorithm for *your* workload?
 - example: sorting an almost sorted collection

Take-Home Lessons (cont.)

- Use the tools in your toolbox!
 - interfaces, generic data structures
 - lists/stacks/queues, trees, heaps, hash tables
 - recursion, recursive backtracking, divide-and-conquer
- Use built-in/provided collections/interfaces:
 - `java.util.ArrayList<T>` (implements `List<T>`)
 - `java.util.LinkedList<T>` (implements `List<T>` and `Queue<T>`)
 - `java.util.Stack<T>`
 - `java.util.TreeMap<K, V>` (a balanced search tree)
 - `java.util.HashMap<K, V>` (a hash table)
 - `java.util.PriorityQueue<T>` (a heap)

} implement `Map<K, V>`
- But use them intelligently!
 - ex: `LinkedList` maintains a reference to the last node in the list
 - `list.add(item, n)` will add `item` to the end in $O(n)$ time
 - `list.addLast(item)` will add `item` to the end in $O(1)$ time!